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NEW

Department of Examinations – Sri Lanka

G.C.E. (A/L) Examination – 2019

7 – Mathematics I

New Syllabus

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included.

1. Let $A = \{x \in \mathbb{R}: |x-2| \geq 2\}$ and $B = \{x \in \mathbb{R}: |x-1| < 3\}$ be subsets of $\mathbb{R}$. Find $A \cap B$ and $A \cup B'$.

$A = \{x : |x-2| \geq 2\}$

Since, $|x-2| \geq 2 \Rightarrow (x-2) \leq -2$ or $(x-2) \geq 2 \Rightarrow x \leq 0$ or $x \geq 4$, we have

$A = (-\infty, 0] \cup [4, \infty)$. 5

$B = \{x : |x-1| < 3\}$

Since $|x-1| < 3 \Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4$, we have $B = (-2,4)$. 5

Thus, $A \cap B = (-2,0]$. 5

$B' = (-\infty, -2] \cup [4, \infty)$. 5

Thus, $A \cup B' = (-\infty, 0] \cup [4, \infty)$. 5

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2. Let A and B be subsets of a universal set S. The set $A \setminus B$ is defined, in the usual notation, by $A \setminus B = A \cap B'$. Show that $A \setminus B = B' \setminus A'$ and $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

$B' \setminus A' = B' \cap (A')'$

$= B' \cap A$ (Double complementation) 5

$= A \cap B'$ (Commutative law)

$= A \setminus B$. 5

$(A \setminus B) \setminus C = (A \setminus B) \cap C'$

$= (A \cap B') \cap C'$ 5

$= A \cap (B' \cap C')$ (Associativity) 5

$= A \cap (B \cup C)'$ (De Morgan's law's)

$= A \setminus (B \cup C)$. 5

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3. Show that the compound propositions $(p \Rightarrow q) \vee (p \Rightarrow r)$ and $p \Rightarrow (q \vee r)$ are logically equivalent.

p	q	r	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$	$q \vee r$	$p \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T
					5	5	5

The two compound propositions $(p \Rightarrow q) \vee (p \Rightarrow r)$ and $p \Rightarrow (q \vee r)$ have the same truth-value for each combination of p, q and r. 5

Therefore, the two compound propositions $(p \Rightarrow q) \vee (p \Rightarrow r)$ and $p \Rightarrow (q \vee r)$ are logically equivalent. 5

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4. Using the method of contrapositive, prove that if n^3+5 is odd, then n is even.

Let p : $n^3 + 5$ is odd, and
q : n is even. 5

Since, $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent, we need to prove:

n is odd $\Rightarrow n^3 + 5$ is even. 5

Let n be odd.

n is odd $\Rightarrow n = 2m + 1$ for some $m \in \mathbb{Z}$. 5

Thus, $n^3 + 5 = (2m+1)^3 + 5 = (8m^3 + 12m^2 + 6m + 1) + 5$

$= 2(4m^3 + 6m^2 + 3m + 3)$ 5

$= 2p$, where $p = 4m^3 + 6m^2 + 3m + 3 \in \mathbb{Z}$

Therefore, $n^3 + 5$ is even. 5

Hence the result.

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5. Solve the simultaneous equations $2\log_9 x + \log_3 y = 3$ and $2^{x+3} - 8^{y+1} = 0$ for x and y.

$2\log_9 x + \log_3 y = 3 \Rightarrow \log_9 x^2 + \log_3 y = 3$ --- (1)

Since $\log_3 x^2 = \log_3 9 \times \log_9 x^2 \Rightarrow \log_3 x^2 = \log_3 3^2 \times \log_9 x^2 \Rightarrow \log_3 x^2 = 2\log_9 x^2$,

from (1) we have $\frac{1}{2}\log_3 x^2 + \log_3 y = 3 \Rightarrow \log_3 x^2 + 2\log_3 y = 6$ 5 5

$\Rightarrow \log_3 x^2 + \log_3 y^2 = 6$

$\Rightarrow \log_3(x^2y^2) = 6$

$\Rightarrow x^2y^2 = 3^6$. --- (2) 5 5

$2^{x+3} - 8^{y+1} = 0 \Rightarrow 2^{x+3} - 2^{3(y+1)} = 0 \Rightarrow x + 3 = 3(y+1) \Rightarrow x = 3y$. --- (3)

From (2) and (3), $(3y)^2y^2 = 3^6 \Rightarrow 3^2y^4 = 3^6 \Rightarrow y = 3$, since $y > 0$. 5

Thus, $x = 9$.

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6. Find all real values of x satisfying the inequality $x \le \frac{2}{x-1}$.

$x \le \frac{2}{x-1} \Rightarrow \frac{2}{x-1} - x \ge 0 \Rightarrow -\frac{(x+1)(x-2)}{(x-1)} \ge 0$. 5

Number line for the problem is as follows: 15

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x \le 2$	$x = 2$	$2 < x$
Sign of $\frac{-(x+1)(x-2)}{(x-1)}$	$\frac{(-)(-)(-)}{(-)} = (+)$	$= 0$	$\frac{(-)(+)(-)}{(-)} = (-)$	*Undefined*	$\frac{(-)(+)(-)}{(+)} = (+)$	$= 0$	$\frac{(-)(+)(+)}{(+)} = (-)$

The solution is the values of x satisfying the inequalities $x \le -1$ or $1 < x \le 2$. 5

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7. Let $f(x)=x^3+1$ and $g(x)=ax+b$ for $x\in\mathbb{R}$, where a and b are real constants. It is given that $f(g(0))=2$ and $g(f(0))=3$. Find the values of a and b.

With these values for a and b, find $g^{-1}(x)$.

$f(x) = x^3 + 1$ and $g(x) = ax + b$ for $x \in \mathbb{R}$, where a and b are real constants.
Thus, $g(0) = b$ and $f(0) = 1$.

$f(g(0)) = 2 \Rightarrow f(b) = 2 \Rightarrow b^3 + 1 = 2 \Rightarrow b^3 = 1 \Rightarrow b = 1$. --- (1) 5
$g(f(0)) = 3 \Rightarrow g(1) = 3 \Rightarrow a + b = 3$. --- (2) 5

From (1) and (2), we have $a = 2$. 5
Thus, $g(x) = 2x + 1$.

Since $g(x)$ is a linear function of x on $\mathbb{R}$, $g(x)$ is a bijection on $\mathbb{R}$. Thus $g^{-1}(x)$ exists on $\mathbb{R}$.
Let $y = g(x) = 2x + 1$.
$y = 2x + 1 \Rightarrow x = \frac{1}{2}(y-1)$. 5

Thus $g^{-1}(x) = \frac{1}{2}(x-1)$. 5

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8. Let $A\equiv(1, 2)$ and $B\equiv(9, 8)$. Find the equation of the perpendicular bisector l of AB.

Two points C and D are taken on l such that $ACBD$ is a square. Show that the area of the square $ACBD$ is 50 square units.

$A \equiv (1,2)$ and $B \equiv (9,8)$.
Let P be the mid-point of line joining the points A and B.
Then $P \equiv \left(\frac{1+9}{2}, \frac{2+8}{2}\right) = (5,5)$. 5

Gradient of the line joining the points A and $B = \frac{8-2}{9-1} = \frac{3}{4}$.
Therefore, the gradient of the perpendicular to $AB = -\frac{4}{3}$. 5

Thus the equation of the perpendicular bisector l of AB is given by $y - 5 = -\frac{4}{3}(x-5)$
$\Rightarrow 4x + 3y = 35$. 5
$AB = [(9-1)^2 + (8-2)^2]^{\frac{1}{2}} = [8^2 + 6^2]^{\frac{1}{2}} = 10$ units. 5

The area of the square $ACBD$ $= 4 \times \left[\frac{1}{2}\left(\frac{1}{2}AB\right) \times \left(\frac{1}{2}AB\right)\right]$
$= 4 \times \left[\frac{1}{2}\left(\frac{1}{2} \times 10\right)\left(\frac{1}{2} \times 10\right)\right]$
$= 50$ Square units. 5

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9. The surface area of a closed rectangular box with a square base of side length x m and height h m is $100\ m^2$. If x is increasing at a rate of $6\ m\ s^{-1}$ while keeping the surface area unchanged, find the rate at which h is changing when $x = 5$ m.

Given that, x is the length of a side of the square base and h is the height of the closed rectangular box.

Surface area of the closed rectangular box $= 2x^2 + 4xh$. 5

Given that $2x^2 + 4xh = 100\ m^2$ and $\frac{dx}{dt} = 6\ m\ s^{-1}$. 5

Differentiating with respect to t,

$$2x^2 + 4xh = 100\ m^2 \Rightarrow 4x\frac{dx}{dt} + 4x\frac{dh}{dt} + 4h\frac{dx}{dt} = 0. \text{ --- (1)}$$ 5

When $x = 5$ m, $2x^2 + 4xh = 100\ m^2 \Rightarrow h = \frac{5}{2}\ m.$ 5

Thus, from (1): $4 \times 5 \times 6 + 4 \times 5 \times \frac{dh}{dt} + 4 \times \frac{5}{2} \times 6 = 0 \Rightarrow \frac{dh}{dt} = -9\ m\ s^{-1}.$ 5

Therefore, the required rate of change $= -9\ m\ s^{-1}$. $\boxed{25}$

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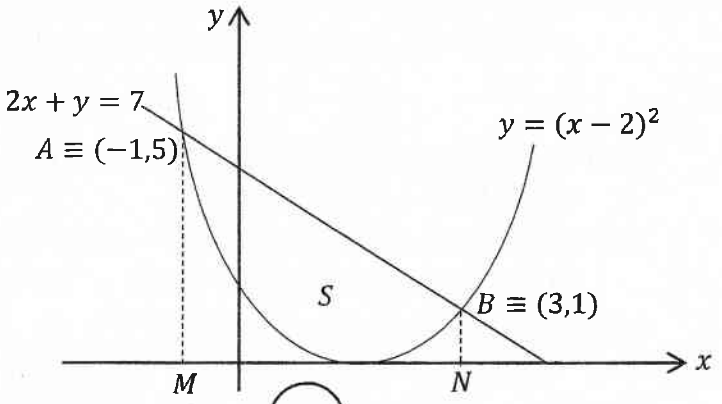
10. Find the area of the region enclosed by the curve $y=(x-2)^2$ and the straight line $2x+y=7$.

Solving the curve $y = (x-2)^2$ and the line $2x + y = 7$ we get:

$7 - 2x = (x-2)^2 \Rightarrow 7 - 2x = x^2 - 4x + 4 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x+1)(x-3) = 0.$

$\Rightarrow x = -1, 3.$

Thus, the curve $y = (x-2)^2$ and the line $2x + y = 7$ meet at the points $A \equiv (-1,5)$ and $B \equiv (3,1)$. 5



Required area $S =$ Area $ABNM - \int_{-1}^{3}(x-2)^2\,dx$ 5

$= \frac{1}{2}(AM + BN) \times MN - \int_{-1}^{3}(x-2)^2$

$= \frac{1}{2}(5+1) \times 4 - \left[\frac{1}{3}(x-2)^3\right]_{-1}^{3} = 12 - \left[\frac{1}{3}(1+27)\right] = \frac{8}{3}$ Square units.

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11. (a) Eighty five students of a certain school have to face two pre-qualifying examinations to qualify for the final examination.

Number of students passed in the first pre-qualifying examination is equal to twice the number of students passed in the second examination. The number of students who passed exactly one examination is 70 and 5 students failed both examinations.

Determine the number of students who passed

(i) each pre-qualifying examination,

(ii) both examinations.

(b) Using truth tables, determine whether each of the following compound propositions is a tautology, a contradiction or neither.

(i) $[p \wedge (\sim q \Rightarrow \sim p)] \Rightarrow q$

(ii) $[p \wedge (p \Rightarrow q)] \wedge (\sim q)$

(iii) $\sim (p \wedge q) \Rightarrow (p \vee q)$

(a) Let E_1 be the set of students who passed the first pre-qualifying examination and E_2 be the set of students who passed the second pre-qualifying examination. 5

Let $n(E_2) = x$ and $n(E_1 \cap E_2) = y$.

Given that, the number of students passed in the first examination is equal to twice the number of students passed in the second examination, i.e.,

$n(E_1) = 2n(E_2) \Rightarrow n(E_1) = 2x$. 5

The number of students who passed exactly one examination is 70 gives

$n(E_1) + n(E_2) - 2n(E_1 \cap E_2) = 70 \Rightarrow 2x + x - 2y = 70.$

$\Rightarrow 3x - 2y = 70.$ --- (1) 10

Since 5 students failed both examinations we have 5

$n(E_1 \cup E_2) = 85 - 5 = 80.$

Therefore, from $n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2)$ we get

$80 = 2x + x - y \Rightarrow 3x - y = 80.$ --- (2) 10

Solving (1) and (2) we get: $x = 30$ and $y = 10$.

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Therefore,

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the number of students passed in the first examination $n(E_1) = 2x = 60$, 10

the number of students passed in the second examination $n(E_2) = x = 30$, and

the number of students passed both examinations $n(E_1 \cap E_2) = y = 10$.

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(b) (i)

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \wedge (\sim q \Rightarrow \sim p)$	$[p \wedge (\sim q \Rightarrow \sim p)] \Rightarrow q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

5 5 5

Since, the truth value of the compound proposition $[p \wedge (\sim q \Rightarrow \sim p)] \Rightarrow q$ is true for all possible combinations of truth values of p and q, 5
$[p \wedge (\sim q \Rightarrow \sim p)] \Rightarrow q$ is a tautology. 5

$\boxed{25}$

(ii)

p	q	$\sim q$	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$[p \wedge (p \Rightarrow q)] \wedge (\sim q)$
T	T	F	T	T	F
T	F	T	F	F	F
F	T	F	T	F	F
F	F	T	T	F	F

5 5 5

Since, the truth value of the compound proposition $[p \wedge (p \Rightarrow q)] \wedge (\sim q)$ is false for all possible combinations of truth values of p and q, 5
$[p \wedge (p \Rightarrow q)] \wedge (\sim q)$ is a contradiction. 5

$\boxed{25}$

(iii)

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee q$	$\sim (p \wedge q) \Rightarrow (p \vee q)$
T	T	T	F	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

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The compound proposition $\sim (p \wedge q) \Rightarrow (p \vee q)$ is neither a tautology nor a contradiction. 10

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12. (*a*) **Using the Principle of Mathematical Induction, prove that**

$$\sum_{r=1}^{n}\left(6r^2-2r-1\right)=n\left(2n^2+2n-1\right) \text{ for all } n\in\mathbb{Z}^+.$$

(*b*) Let $V_r=\dfrac{1}{(r+1)(r+2)}$ for $r\in\mathbb{Z}^+$.

Verify that $V_r=\dfrac{r+1}{r+2}-\dfrac{r}{r+1}$ for $r\in\mathbb{Z}^+$.

Show that $\displaystyle\sum_{r=1}^{n}V_r=\frac{n}{2(n+2)}$ for $n\in\mathbb{Z}^+$.

Also, find $\displaystyle\sum_{r=6}^{16}(2V_r+3)$.

(a) Let $n=1$.

Then $LHS=\sum_{r=1}^{1}(6r^2-2r-1)=6-2-1=3$ and

$RHS=1\cdot(2+2-1)=3.$ 10

5 Thus, the result is true for $n=1$.

Assume that the result is true for $n=p$, i.e.,

$\sum_{r=1}^{p}(6r^2-2r-1)=p(2p^2+2p-1).$ 5

Now consider the case $n=p+1$. 10

$$\begin{aligned}\sum_{r=1}^{p+1}(6r^2-2r-1)&=\sum_{r=1}^{p}(6r^2-2r-1)+\{6(p+1)^2-2(p+1)-1\}\\ &=p(2p^2+2p-1)+\{6(p+1)^2-2(p+1)-1\}\\ &=p\{2(p+1)^2-2p-3\}+\{6(p+1)^2-2(p+1)-1\}\\ &=2(p+1)^2(p+1)-p\{2p+3\}\\ &\qquad+\{4(p+1)^2-2(p+1)-1\}\\ &=2(p+1)^2(p+1)+2(p+1)^2\\ &\qquad+\{-p(2p+3)+2(p+1)^2-2(p+1)-1\}\\ &=2(p+1)^2(p+1)+2(p+1)^2-(p+1).\\ &=(p+1)\{2(p+1)^2+2(p+1)-1\}.\end{aligned}$$

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Thus, the result is true for $n=p+1$, when the result is true for $n=p$. 5

Therefore, from the principle of Mathematical Induction, result is true for all $n\in\mathbb{Z}^+$.

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(b) Given that $V_r=\frac{1}{(r+1)(r+2)}$ for $r\in\mathbb{Z}^+$.

Now consider $\frac{r+1}{r+2}-\frac{r}{r+1}$.

$$\frac{r+1}{r+2}-\frac{r}{r+1}=\frac{(r+1)^2-r(r+2)}{(r+1)(r+2)}=\frac{(r^2+2r+1)-(r^2+2r)}{(r+1)(r+2)}=\frac{1}{(r+1)(r+2)}=V_r.$$ 10

Thus, $V_r = \frac{r+1}{r+2} - \frac{r}{r+1}$ for $r \in \mathbb{Z}^+$.

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When $r = 1$, $V_1 = \frac{2}{3} - \frac{1}{2}$

$r = 2$, $V_2 = \frac{3}{4} - \frac{2}{3}$ 15

$r = 3$, $V_2 = \frac{4}{5} - \frac{3}{4}$

$\vdots \qquad \vdots$

$r = n-2$ $V_{n-2} = \frac{n-1}{n} - \frac{n-2}{n-1}$

$r = n-1$ $V_{n-1} = \frac{n}{n+1} - \frac{n-1}{n}$ 15

$r = n$ $V_n = \frac{n+1}{n+2} - \frac{n}{n+1}$.

Adding n equations, we get $\sum_{r=1}^{n} V_r = \frac{n+1}{n+2} - \frac{1}{2} = \frac{2(n+1)-(n+2)}{2(n+2)} = \frac{n}{2(n+2)}$ for $n \in \mathbb{Z}^+$.

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$\sum_{r=6}^{16}(2V_r + 3) = 2\sum_{r=6}^{16} V_r + \sum_{r=6}^{16}(3)$

$= 2\{\sum_{r=1}^{16} V_r - \sum_{r=1}^{5} V_r\} + 3 \cdot 11$

5 5 5

$= 2\left[\frac{16}{16+2} - \frac{5}{5+2}\right] + 33$ 5 5

$= 2\left[\frac{16}{18} - \frac{5}{7}\right] + 33$

$= 33 + \frac{22}{63}$. 5

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13.(a) Let $a \in \mathbb{R}$. Show that the roots of the equation $x^2+ax-1=0$ are real and distinct.

Let α and β be these roots. Find the quadratic equation that has $2\alpha+1$ and $2\beta+1$ as its roots.

(b) Let $f(x)=x^3+3x^2+px+q$, where p and q are real numbers.

The remainder when $f(x)$ is divided by $(x-1)$ is -12 and $(x-2)$ is a factor of $f(x)$. Find the values of p and q.

Also, find the other linear factors of $f(x)$.

(a) Since $a^2 - 4 \times 1 \times (-1) = a^2 + 4 > 0$ for all $a \in \mathbb{R}$, the roots of the equation $x^2 + ax - 1 = 0$ are real and distinct. 10 5 15

Since α and β are the roots of the equation $x^2 + ax - 1 = 0$, we get $\alpha + \beta = -a$ and $\alpha\beta = -1$. 5 5

Let $A = 2\alpha + 1$ and $B = 2\beta + 1$.

Then, $A + B = (2\alpha + 1) + (2\beta + 1) = 2(\alpha + \beta) + 2 = 2 \times (-a) + 2 = 2(1 - a)$.

$\Rightarrow A + B = 2(1 - a)$. 5 5 5

$AB = (2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2(\alpha + \beta) + 2 = 4 \cdot (-1) + 2 \cdot (-a) + 2$.

5 $\Rightarrow AB = -2(a + 1)$.

Therefore, the required quadratic equation is $x^2 - (A + B)x + AB = 0$. 5

That is, $x^2 - 2(1 - a)x - 2(a + 1) = 0$. 10 45

(b) Let $f(x) = x^3 + 3x^2 + px + q$ ---- (1)

Also, let $f(x) = (x - 1)q_1(x) - 12$ ---- (2) and $f(x) = (x - 2)q_2(x)$ ---- (3). 10

Now, (2) $\Rightarrow f(1) = (1 - 1)q_1(1) - 12 \Rightarrow f(1) = -12$ 5

(3) $\Rightarrow f(2) = (2 - 2)q_2(2) \Rightarrow f(2) = 0$. 5

Thus, from (1):

10 $\left.\begin{aligned} f(1) = -12 &\Rightarrow p + q = -16 \\ f(2) = 0 &\Rightarrow 2p + q = -20 \end{aligned}\right\} \Rightarrow p = -4, q = -12$. 10 40

Now $f(x) = x^3 + 3x^2 - 4x - 12$. --- (4) 5

Since, $f(x)$ is a cubic polynomial with the coefficient of x^3 equals 1, we can write

$f(x) = (x - 2)(x - a)(x - b)$, --- (5) 10

where a and b are the remaining roots of $f(x) = 0$.

Comparing coefficient of x^0 in equations (4) and (5) we get

$-2ab = -12 \Rightarrow ab = 6$ --- (6) 10

Comparing coefficient of x^2 in equations (4) and (5) we get

$$-2-(a+b)=3 \Rightarrow a+b=-5 \text{ --- } (7) \quad 5$$

Solving (6) and (7) we get $a=-2, b=-3$ or $a=-3, b=-2.$ 10

Therefore, $f(x)=(x-2)(x+2)(x+3).$ 10

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14. (a) Let $k \in \mathbb{R}$. The coefficients of x^{20} and x^{21} in the binomial expansion of $(1 + kx)^{23}$ are equal. Show that $k = 7$.

(b) By discarding the terms involving powers of x greater than 3, find an approximate value for $(1.7)^{23} + (0.3)^{23}$.

(c) A person opened a bank account by depositing Rs.50000 at the beginning of a month. He then deposited Rs.20000 at the beginning of every month for two years. The account pays 0.5% interest compounded monthly. Find the balance in the account after two years.

At the end of every month after this two year period, he withdrew Rs. 20000 from the account. For how long will there be money left in the account for him to continuously withdraw Rs.20000 per month?

(a) Let $k \in R$.

$$(1+kx)^{23} = \sum_{r=0}^{23} {}^{23}C_r (kx)^{23-r}$$ 10 5

Coefficient of x^{20} is obtained when $23 - r = 20$ or when $r = 3$.

Coefficient of $x^{20} = {}^{23}C_3\ k^{20}$. 5

Coefficient of $x^{21} = {}^{23}C_2\ k^{21}$. 5

Equating the coefficients of x^{20} and x^{21}, we get ${}^{23}C_3\ k^{20} = {}^{23}C_2\ k^{21}$ 5

$\Rightarrow \frac{23!}{3! \times 20!} = \frac{23!}{2! \times 21!}\ k.$ 5

Thus, $k = \frac{21}{3} = 7$. 5

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(b) $(1.7)^{23} = (1 + 7 \times 0.1)^{23}$ 5

$$= \sum_{r=0}^{23} {}^{23}C_r\ 7^{23-r}\ (0.1)^{23-r}$$

$$\cong {}^{23}C_{23}\ 7^0\ (0.1)^0 + {}^{23}C_{22}\ 7^1\ (0.1)^1 + {}^{23}C_{21}\ 7^2\ (0.1)^2 + {}^{23}C_{20}\ 7^3\ (0.1)^3,$$

(ignoring powers of 3 and higher, in 0.1) 5

$(0.3)^{23} = (1 + 7 \times (-0.1))^{23}$ 5

$$= \sum_{r=0}^{23} {}^{23}C_r\ 7^{23-r}\ (-0.1)^{23-r}$$ 5

$$\cong {}^{23}C_{23}\ 7^0\ (0.1)^0 - {}^{23}C_{22}\ 7^1\ (0.1)^1 + {}^{23}C_{21}\ 7^2\ (0.1)^2 - {}^{23}C_{20}\ 7^3\ (0.1)^3,$$

(ignoring powers of 3 and higher, in 0.1)

Adding the two, we find

$$(1.7)^{23} + (0.3)^{23} = 2 + 2 \times {}^{23}C_{21}\ 7^2\,(0.1)^2 \quad 5$$

$$= 2 + 2 \times \frac{23!}{21! \times 2!} \times 49 \times 0.01$$

$$= 2 + 23 \times 22 \times 0.49$$

$$= 2 + 247.94$$

$$= 249.94. \quad 5$$

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(c) Let $A = 50{,}000, B = 20{,}000$ and $r = \frac{0.5}{100} = 0.005.$

Month	Amount in the account at the end of the month
1	$A(1+r)$ 5
2	$A(1+r)^2 + B(1+r)$ 5
3	$A(1+r)^3 + B(1+r)^2 + B(1+r)$ 5
4	$A(1+r)^4 + B(1+r)^3 + B(1+r)^2 + B(1+r)$
5	$A(1+r)^5 + B(1+r)^4 + B(1+r)^3 + B(1+r)^2 + B(1+r)$
⋮	⋮

Value in the account at the end of 24 months

$$= A(1+r)^{24} + B(1+r)(1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{22}) \quad 15$$

$$= A(1+r)^{24} + B(1+r)\frac{((1+r)^{23} - 1)}{(1+r-1)} \quad 5$$

$$= A(1+r)^{24} + \frac{B(1+r)}{r} \times ((1+r)^{23} - 1).$$

$$= \left(A + \frac{B}{r}\right)(1+r)^{24} - \frac{B(1+r)}{r} \quad 5$$

$$= \left(50000 + \frac{20000}{0.005}\right)(1.005)^{24} - \frac{20000(1.005)}{0.005} \quad 5$$

$$= 405 \times 10^4 \times (1.005)^{24} - 402 \times 10^4$$

$$\cong 544998. \quad 10$$

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Now suppose the person starts withdrawing Rs.20000/- each month.

Let $C = 544998$.

Month (after 24 months)	Amount in the account after withdrawal
1	$C(1+r) - B$
2	$C(1+r)^2 - B(1+r) - B$
3	$C(1+r)^3 - B(1+r)^2 - B(1+r) - B$
4	$C(1+r)^4 - B(1+r)^3 - B(1+r)^2 - B(1+r) - B$
$\vdots$	$\vdots$

Need largest $k \in N$ such that:

$$C(1+r)^k - B(1+r)^{k-1} - B(1+r)^{k-2} - \cdots - B(1+r) - B \geq 0.$$

That is $C(1+r)^k - \frac{B}{r}((1+r)^k - 1) \geq B.$ 5 10

$\Rightarrow (B - Cr)(1+r)^k \leq B.$ 5

$\Rightarrow (1+r)^k \leq \frac{B}{B-Cr}$

$\Rightarrow (1.005)^k \leq \frac{20000}{20000 - 544998 \times 0.005} = 1.158$

$\Rightarrow k \leq \frac{\log_{10} 1.158}{\log_{10} 1.005} = 29.4.$

Therefore, $k = 29$ months. 5

He can withdraw Rs. 20,000 per month for 29 months.

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15. Show that there are two straight lines l_1 and l_2 passing through the point $(-2, 8)$ and sum of the intercepts on the axes is 6.

A straight line meets the above two straight lines l_1 and l_2 at points P and Q respectively. If the midpoint of the line segment PQ is $(1, 5)$, find the equation of the line PQ.

Show that the equation of the straight line passing through the point of intersection of straight lines l_1 and l_2, and perpendicular to PQ is $4y = x + 34$.

Let the equation of straight line be $\frac{x}{a} + \frac{y}{b} = 1$. 5

Since the sum of the intercepts is 6,

$a + b = 6, \Rightarrow a = 6 - b$ --- (1). 5

Since the point $(-2,8)$ is on the straight line,

$\frac{-2}{a} + \frac{8}{b} = 1 \Rightarrow 8a - 2b = ab.$ 5

By (1), $8(6 - b) - 2b = (6 - b)b$ 5

$\Rightarrow 48 - 8b - 2b = 6b - b^2$

$\Rightarrow b^2 - 16b + 48 = 0$

$\Rightarrow (b - 4)(b - 12) = 0$ 10

$\Rightarrow b = 4, 12.$

When $b = 4$, $a = 2$ and when $= 12$, $a = -6$. 10

So there are two straight lines and corresponding equations are 5

$l_1: \frac{x}{2} + \frac{y}{4} = 1 \Rightarrow 2x + y = 4 \Rightarrow y + 2x - 4 = 0$ --- (2)

$l_2: \frac{x}{-6} + \frac{y}{12} = 1 \Rightarrow -2x + y = 12 \Rightarrow y - 2x - 12 = 0$ --- (3) 5

$\boxed{50}$

Aliter

Let the equation of straight line be $y - 8 = m(x + 2)$. 10

The x –interpret is given by $x = \frac{-(2m+8)}{m}$. 5

The y –intercept is given by $y = 2m + 8$. 5

It is given that, $\frac{-(2m+8)}{m} + (2m + 8) = 6$. 10

Thus, $\frac{-(2m+8)}{m} + (2m + 8) = 6 \Rightarrow -2m - 8 + 2m^2 + 8m = 6m \Rightarrow m^2 - 4 = 0$

$\Rightarrow m = \pm 2.$ 10

So there are two straight lines and corresponding equations are

$y - 8 = 2(x + 2) \Rightarrow y - 2x - 12 = 0$ and 5

$y - 8 = -2(x + 2) \Rightarrow y + 2x - 4 = 0$ 5

$\boxed{50}$

Let the gradient of the straight line PQ be m and (x, y) be any point on PQ.
Since $(1,5)$ on PQ,

$\frac{y-5}{x-1} = m \Rightarrow y = mx + (5 - m)$. 10

Since l_1 and PQ intersect at P, by (2),

$mx + (5 - m) + 2x - 4 = 0 \Rightarrow (2 + m)x = m - 1 \Rightarrow x = \frac{m-1}{m+2}$. 10

So, $y = m\frac{m-1}{m+2} + (5 - m) = \frac{m^2-m+3m+10-m^2}{m+2} = \frac{2m+10}{m+2}$. 10

$\therefore P \equiv \left(\frac{m-1}{m+2}, \frac{2m+10}{m+2}\right)$.

Since l_2 and PQ intersect at$Q$, by (3),

$mx + (5 - m) - 2x - 12 = 0 \Rightarrow (m - 2)x = (m + 7) \Rightarrow x = \frac{m+7}{m-2}$. 10

So, $y = m\frac{m+7}{m-2} + (5 - m) = \frac{m^2+7m+7m-10-m^2}{m-2} = \frac{14m-10}{m-2}$. 10

$\therefore Q \equiv \left(\frac{m+7}{m-2}, \frac{14m-10}{m-2}\right)$.

Since $(1,5)$ is the midpoint of PQ, $\frac{1}{2}\left(\frac{m-1}{m+2} + \frac{m+7}{m-2}\right) = 1$ 10

$\Rightarrow (m - 1)(m - 2) + (7 + m)(m + 2) = 2(m^2 - 4)$

$\Rightarrow m^2 - 3x + 2 + m^2 + 9m + 14 = 2m^2 - 8$

$\Rightarrow 6m = -24$

$\Rightarrow m = -4$. 10

$\therefore$ The equation of PQ is $y = -4x + (5 - (-4))$. i.e. $y + 4x = 9$. 10

$\boxed{80}$

The gradient of a straight line perpendicular to PQ is $\frac{1}{4}$. 10

The equation of the straight line passing through $(-2,8)$ and perpendicular to PQ is

$\frac{y-8}{x-(-2)} = \frac{1}{4} \Rightarrow 4y - 32 = x + 2 \Rightarrow 4y = x + 34$. 10

$\boxed{20}$

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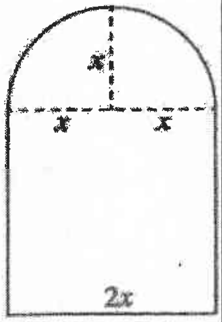
16. (a) Find $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^3 - a^3}$.

(b) Differentiate each of the following with respect to x.

(i) $\ln\left(x + e^{\sqrt{x}}\right)$ (ii) $\left(x + \sqrt{x^2 + a^2}\right)^3$ (iii) $\sqrt{\dfrac{1+e^x}{1-e^x}}$

(c) A window is in the shape of a rectangle surmounted by a semicircle as shown in the figure. The entire perimeter of the window is $(\pi + 4)$ m. By taking x m as the radius of the semicircle, show that the area of the window A m^2 is given by $A = k\left(2x - x^2\right)$, where $k = \frac{1}{2}(\pi + 4)$.

Find the value of x such that the area of the window is maximum.



(a) $\lim\limits_{x\to a} \frac{x^2-a^2}{x^3-x^3} = \lim\limits_{x\to a} \frac{(x-a)}{(x-a)(x^2+ax+a^2)} = \lim\limits_{x\to a} \frac{1}{(x^2+ax+a^2)} = \frac{1}{3a^2}$. 5

10 10

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(b) (i) Let $y = \ln(x + e^{\sqrt{x}})$.

$\frac{dy}{dx} = \frac{1}{x+e^{\sqrt{x}}} \cdot \left\{1 + e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}}\right\} = \frac{(2\sqrt{x}+e^{\sqrt{x}})}{2\sqrt{x}(x+e^{\sqrt{x}})}$. 5

5 5 10

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(ii) Let $y = (x + \sqrt{x^2 + a^2})^3$.

$\frac{dy}{dx} = 3(x + \sqrt{x^2 + a^2})^2 \left\{1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \cdot 2x\right\}$. 5 + 10

$\Rightarrow \frac{dy}{dx} = 3(x + \sqrt{x^2 + a^2})^2 \left\{1 + \frac{x}{\sqrt{x^2+a^2}}\right\}$

$\Rightarrow \frac{dy}{dx} = \frac{3(x+\sqrt{x^2+a^2})^3}{\sqrt{x^2+a^2}}$. 10

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(iii) Let $y = \sqrt{\frac{1+e^x}{1-e^x}}$.

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{-\frac{1}{2}} \cdot \left[\frac{(1-e^x)\cdot(0+e^x)-(1+e^x)\cdot(0-e^x)}{(1-e^x)^2}\right]$ 5 + 10

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{-\frac{1}{2}} \cdot \left[\frac{2e^x}{(1-e^x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-e^x}{1+e^x}\right)^{\frac{1}{2}} \cdot \left[\frac{e^x}{(1-e^x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{\sqrt{1+e^x}\cdot(1-e^x)^{\frac{3}{2}}}.$$ 10

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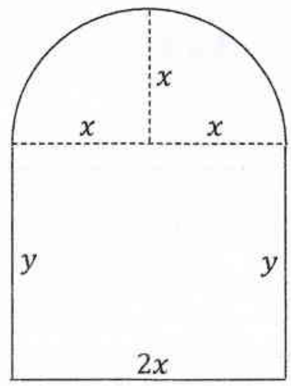
(c) Let y be the height of the window.

The entire perimeter of the window $= \pi x + 2x + 2y$. 5

Given that $\pi x + 2x + 2y = \pi + 4$. 5

Thus $\pi x + 2x + 2y = \pi + 4 \Rightarrow y = \frac{1}{2}[\pi + 4 - (\pi + 2)x]$.

The area of the window $A = \frac{1}{2}\pi x^2 + 2xy$. 5 5



Thus, $A = \frac{1}{2}\pi x^2 + 2x\frac{1}{2}[\pi + 4 - (\pi + 2)x]$ 5

$= \frac{1}{2}\pi x^2 + (\pi + 4)x - (\pi + 2)x^2$

$= \frac{1}{2}(\pi+4)(2x - x^2)$

$= k(2x - x^2)$, where $k = \frac{1}{2}(\pi + 4)$. 5

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5 5

$$\frac{dA}{dx} = k(2 - 2x) = 2k(1 - x)\begin{cases} > 0 & \text{if } x < 1 \\ = 0 & \text{if } x = 1 \\ < 0 & \text{if } x > 1 \end{cases}, \text{ where } k = \frac{1}{2}(\pi + 4) > 0.$$

Thus, the area of the window A is maximum when $x = 1$. 5

Therefore, the maximum area of the window $= \frac{1}{2}(\pi + 4)(2 \cdot 1 - 1 \cdot 1)$

$= \frac{1}{2}(\pi + 4)\ m^2$. 5

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17. (a) Using the method of integration by parts, evaluate $\int (x+1)^2 e^x \, dx$.

(b) The following table gives the values of the function $f(x)=\frac{1}{(2-x)^2}$ correct to four decimal places, for values of x between 0 and 1 at intervals of length 0.2.

x	0.00	0.20	0.40	0.60	0.80	1.00
$f(x)$	0.2500	0.3086	0.3906	0.5102	0.6944	1.0000

Using Trapezoidal rule, find an approximate value for $I=\int_0^1 \frac{1}{(2-x)^2}dx$, correct to three decimal places.

Find I using the substitution $u=2-x$ or otherwise and compare with the approximation obtained above.

(a) $\int (x+1)^2 e^x \, dx = \int (x+1)^2 \frac{de^x}{dx} dx$ 10

$= (x+1)^2 e^x - \int e^x \frac{d}{dx}(x+1)^2 \, dx$ 10

$= (x+1)^2 e^x - 2\int e^x (x+1) \, dx$ 10

10 $= (x+1)^2 e^x - 2\int \frac{de^x}{dx}(x+1) \, dx$

$= (x+1)^2 e^x - 2\left[e^x(x+1) - \int e^x \frac{d}{dx}(x+1) \, dx\right]$ 10

$= (x+1)^2 e^x - 2[e^x(x+1) - \int e^x \, dx]$ 10

$= (x+1)^2 e^x - 2[e^x(x+1) - e^x] + C.$ 10

$\boxed{70}$

(b) Let $h = \frac{1-0}{5} = 0.20.$ 10

Using Trapizoidal rule,

$I \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5]$ 10

10 $= \frac{0.20}{2}[0.2500 + 2 \times 0.3086 + 2 \times 0.3906 + 2 \times 0.5102 + 2 \times 0.6944 + 1.000]$

$= \frac{0.20}{2}[0.2500 + 0.6172 + 0.7812 + 1.0204 + 1.3888 + 1.0000]$ 20

$= 0.1 \times 5.0576$

$= 0.5058.$ 10

$\boxed{60}$

$$\int_0^1 \frac{1}{(2-x)^2}dx = -\int_2^1 \frac{1}{u^2}du = -\left[-\frac{1}{u}\right]_2^1 = -\left[-1-\left(-\frac{1}{2}\right)\right] = \frac{1}{2}. \quad 5$$

10

Therefore, the error $= |Actual\ value - Estimated\ value| = |0.5000 - 0.5058|$

5 $= 0.0058.$

$\boxed{20}$

Confidential

NEW

Department of Examinations – Sri Lanka

G.C.E. (A/L) Examination – 2019

7 – Mathematics II

New Syllabus

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included.

1. Let $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$, where a, b and c are distinct non-zero real constants. If $\Delta = 0$, show that $abc = -1$.

$$\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad 5$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \Rightarrow \Delta = (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ --- (1)} \quad 5$$

Now, $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow{R_2 \to R_2 + (-1)R_1, R_3 \to R_3 + (-1)R_1} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad 5$$

$$= (b-a)(c^2-a^2) - (b^2-a^2)(c-a)$$
$$= (a-b)(b-c)(c-a). \quad 5$$

Thus from (1): $\Delta = (1+abc)(a-b)(b-c)(c-a)$.

Since a, b and c are distinct real numbers, $(a-b)(b-c)(c-a) \neq 0$ and hence

$\Delta = 0 \Rightarrow 1 + abc = 0 \Rightarrow abc = -1.$ 5

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2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 4 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$. Find $A+B$, AC and BC. Verify that $(A+B)C = AC + BC$.

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 4 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$.

$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 2 \\ -1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}.$ 5

$AC = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 1 \\ 9 & 5 \end{pmatrix}.$ 5

$BC = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 12 & -3 \end{pmatrix}.$ 5

$(A+B)C = \begin{pmatrix} 3 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 2 \\ 21 & 2 \end{pmatrix}.$ --- (1) 5

$AC + BC = \begin{pmatrix} 14 & 1 \\ 9 & 5 \end{pmatrix} + \begin{pmatrix} 11 & 1 \\ 12 & -3 \end{pmatrix} = \begin{pmatrix} 25 & 2 \\ 21 & 2 \end{pmatrix}.$ --- (2) 5

From (1) and (2): $(A+B)C = AC + AB$.

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3. The time X taken (in hours) for assembling a motorcycle follows a normal distribution with mean μ and standard deviation 5. If 10% of the motorcycles are assembled in less than 14 hours, find the mean μ.

Since 10% of the motor cycles are assembled in less than 14 hours,

$$P(Z < z) = 0.1000$$ 5

By normal distribution table, we have,

$$z = -1.2816.$$ 5

Since $X = 14$ and $\sigma = 5$ are given and

$$z = \frac{X-\mu}{\sigma} \Rightarrow -1.2816 = \frac{14-\mu}{5}$$ 5

$$\Rightarrow \mu = 14 + 5 \times 1.2816$$

$$\Rightarrow \mu = 20.408.$$

The mean of the distribution is 20.408 hours. 5

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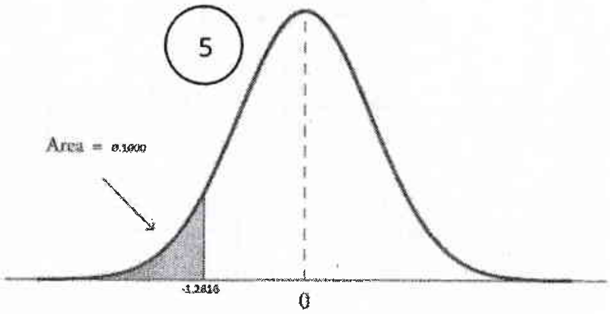

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4. A company has two sections A and B with 50 and 60 employees in each, respectively. In a particular year the average and the standard deviation of the monthly wages in the two sections are given in the following table:

Section	Number of employees	Average monthly wages (Rs.)	Standard deviation of monthly wages (Rs.)
A	50	40 000	6 750
B	60	35 000	7 000

Determine which section has the larger variability in wages.

Coefficient of Variance of wages for Section A $(CV_A) = \frac{\sigma}{\mu} \times 100$ 5

$$= \frac{6750}{40000} \times 100$$

$$= 16.875\%.$$ 5

Coefficient of Variance of wages for Section B $(CV_B) = \frac{\sigma}{\mu} \times 100$

$$= \frac{7000}{35000} \times 100$$

$$= 20.000\%.$$ 5

Since $CV_B > CV_A$, there is a greater variability in wages of Section B than that of section A.

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5. The sum of the numbers and the sum of the squares of the numbers of a set of 20 observations are 140 and 2260 respectively.
   (i) Find the mean and the standard deviation of the 20 observations.
   (ii) If the median is 10, find the coefficient of skewness and comment on the shape of the distribution of the set of 20 observations.

(i) $\sum x_i = 140, \sum x_i^2 = 2260, n = 20$ 5

Mean $\mu = \frac{\sum x_i}{n} = \frac{140}{20} = 7.$ 5

Standard Deviation $\sigma = \sqrt{\frac{\sum x_i^2 - n\mu^2}{n}} = \sqrt{\frac{2260-20\times7^2}{20}} = \sqrt{64} = 8.$ 5

(ii) Coefficient of Skewness $= \frac{3(Mean-median)}{\sigma} = \frac{3\times(7-10)}{8} = -1.125.$ 5

Since Coefficient of Skewness < 0, the data set is negatively skewed. 5

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6. The probability that a randomly selected seed from a packet germinates is 0.7. If five seeds are randomly selected from the packet for planting, find the probability that
   (i) at least one of the seeds will germinate,
   (ii) exactly three seeds will germinate.

Let X denote the number of seeds that will germinate.

Then $X \sim \text{binomial}(n = 5, p = 0.7)$ 5

(i) P (at least one of the seeds will germinate) =1- P (none of the seeds will germinate)

$= 1 - P(X = 0)$ 5

$= 1 - {}^5C_0 \times 0.7^0 \times 0.3^5$

$= 1 - 0.00243$

$=0.998.$ 5

(ii) P(exactly three of the seeds will germinate)

$= P(X = 3)$ 5

$= {}^5C_3 \times 0.7^3 \times 0.3^2$

$= 10 \times 0.343 \times 0.09 = 0.343\times 0.9$

$= 0.031.$ 5

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7. A box has two red pens, two blue pens and a black pen. Two pens are randomly selected without replacement. Find the probability that both pens selected are of

(i) the same colour,

(ii) different colours.

Let X be the color of 1st pen and Y be the color of 2nd pen.

(i) P(Both selected pen are of the same color)

5 $= P(X = blue, Y = blue) + P(X = red, Y = red) + P(X = black, Y = black)$

5 $= P(Y = Blue|\, X = blue)P(X = blue) + P(Y = red|X = red)P(X = red) + P(Y = black|X = black)P(X = black)$

$= \frac{1}{4} \times \frac{2}{5} + \frac{1}{4} \times \frac{2}{5} + \frac{0}{4} \times \frac{1}{5} = \frac{4}{20} = \frac{1}{5}.$ 5

(ii) P(Both pens selected are of different colours) + P(Both pens selected are of the same colour) = 1. 5

P (Both pens selected are of different colors) = $1 - \frac{1}{5} = \frac{4}{5}.$ 5

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8. The probability mass function of a discrete random variable X is given below:

x	0	1	2	3
$P(X = x)$	0.2	0.2	0.3	0.3

Find $E(X)$.

Let Y be the random variable given by $Y = 2X - 3$. Find $E(Y)$ and the probability that Y is positive.

$E(X) = \sum_x xP(X = x)$

$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3)$ 5

$= 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.3$

$= 0 + 0.2 + 0.6 + 0.9$

$= 1.7$ 5

Given $Y = 2X - 3$.

$E(Y) = E(2X - 3) = E(2X) - E(3) = 2E(X) - 3 = 2 \times 1.7 - 3 = 0.4$. 5

The probability that Y is positive $= P(Y > 0)$

$= P(2X - 3 > 0)$

$= P(X > 1.5)$ 5

$= P(X = 2) + P(X = 3)$

$= 0.3 + 0.3 = 0.6.$ 5

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9. Suppose that A and B are exhaustive events of a sample space S. If $P(A)=\frac{2}{3}$ and $P(A\cap B)=\frac{1}{5}$, find (i) $P(B)$, (ii) $P(A|B)$, (iii) $P(A'|B')$.

(i) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow 1=\frac{2}{3}+P(B)-\frac{1}{5}$ (Since A and B are exhaustive events) 5

$\Rightarrow P(B)=1-\frac{7}{15}=\frac{8}{15}$. 5

(ii) $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{5}}{\frac{8}{15}}=\frac{3}{8}$. 5

(iii) $P(A'|B')=\frac{P(A'\cap B')}{P(B')}=\frac{P((A\cup B)')}{1-P(B)}=\frac{1-P(A\cup B)}{1-P(B)}=\frac{1-1}{1-\frac{8}{15}}=0.$ 5

5

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10. Let X be a continuous random variable with probability density function $f(x)$ given by

$$f(x)=\begin{cases} k(3x-1), & 1\le x\le 4,\\ 0, & \text{otherwise,}\end{cases}$$

where k is a positive constant.

Find

(i) the value of k,

(ii) the mean of X.

(i) Since f is probability density function of X,

$$\int_{-\infty}^{\infty} f(x)dx=1. \quad 5$$

$\Rightarrow \int_{-\infty}^{1} f(x)dx+\int_{1}^{4} f(x)dx+\int_{4}^{\infty} f(x)dx=1$ 5

$\Rightarrow 0+\int_{1}^{4} k(3x-1)dx+0=1$

$\Rightarrow k\left[\left(\frac{3x^2}{2}-x\right)\right]_1^4=1$

$\Rightarrow k\times\frac{39}{2}=1$

$\Rightarrow k=\frac{2}{39}$ 5

(ii) Mean of $X=E(X)=\int_{-\infty}^{\infty} xf(x)dx$ 5

$\Rightarrow E(X)=\int_{-\infty}^{1} xf(x)dx+\int_{1}^{4} xf(x)dx+\int_{4}^{\infty} xf(x)dx=\int_{1}^{4} kx(3x-1)dx$ 5

$\Rightarrow E(X)=\int_{1}^{4} k(3x^2-x)dx=k\times\left[\frac{3x^3}{3}-\frac{x^2}{2}\right]_1^4=\frac{2}{39}\times\frac{111}{2}=\frac{37}{13}=2.8462.$

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11. A company owns two machines A and B which have different production capacities for high, medium and low grade nails. This company should produce at least 7, 6 and 13 tons per week of high, medium and low grade nails respectively to meet the demand of the market. It costs the company Rs. 10000 and Rs. 8000 per day to operate machines A and B respectively.

The following table gives the capacity of production in tons per day of each machine on each grade of nails.

Grade of nails	Capacity (tons/day)	
	A	B
High	2	1
Medium	1	1
Low	2	3

The company wishes to find out the number of days that each machine be operated per week to minimize the total production cost, while meeting the demand.

(i) Formulate this as a linear programming problem.

(ii) Sketch the feasible region.

(iii) Using the graphical method, find the solution of the problem formulated in part (i) above.

(iv) Due to a technical issue, the machine B has to be operated for a week at most twice as many days as the machine A is operated.

Find the increase of total production cost per week, if the company still wishes to minimize the production cost.

(i) Let x be the number of days per week machine A is operated and y be the number of days per week machine B is operated.

Linear programming problem is

$minimize \quad z = 1000x + 8000y$ 10

$Subject\ to$

10 $2x + y \geq 7$ 10

$x + 6 \geq 6$

10 $2x + 3y \geq 13$

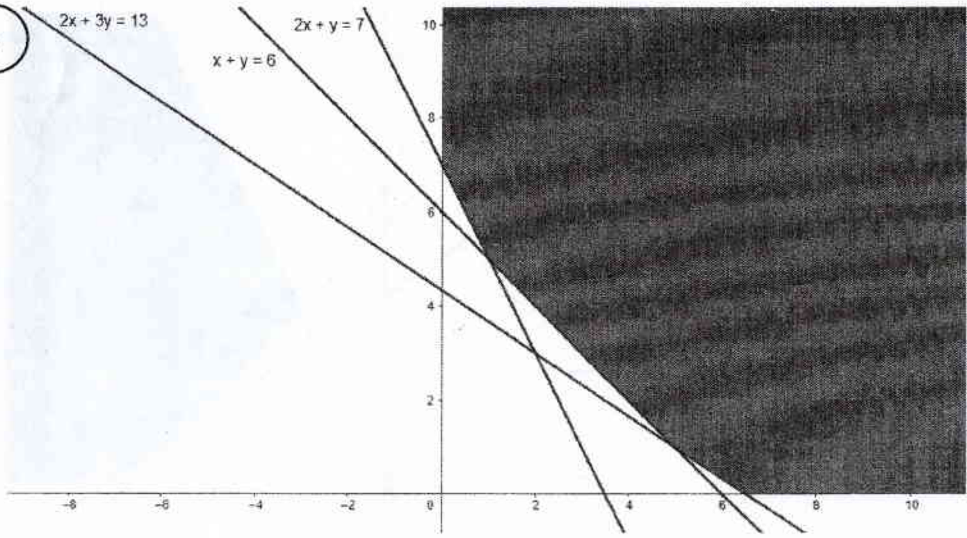
$x \geq 0, y \geq 0$ 10

$\boxed{50}$

(ii)

5 5

5



10

25

(iii)

Point	Value of z
$P_1 \equiv (0,7)$	56, 000
$P_2 \equiv (1,5)$	50, 000
$P_3 \equiv (5,1)$	58, 000
$P_4 \equiv (6.5,0)$	65, 000

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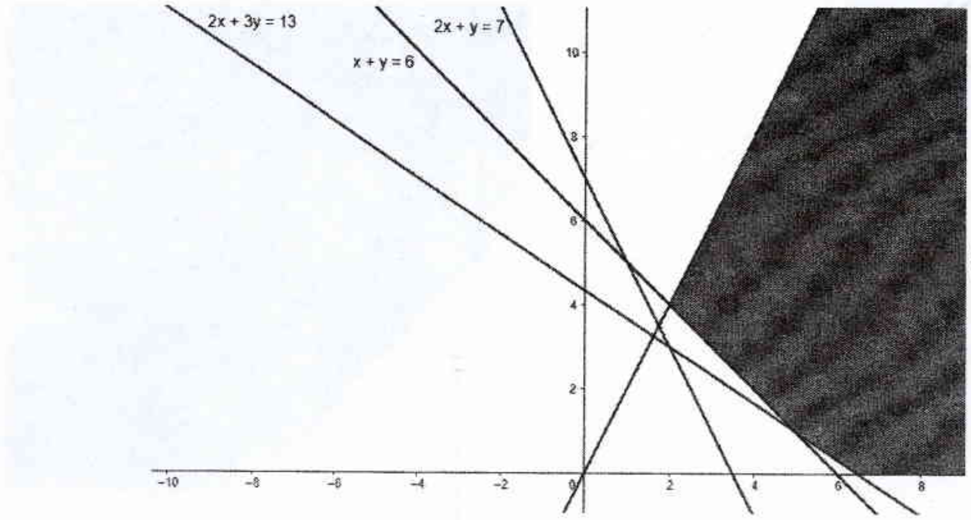
Minimum $z = 50,000$ at $P_2 \equiv (1,5)$. 5

Machine A should be operated 1 days per week and machine B should be operated 5 days per week. 10

35

(iv) With the new constraint $y \leq 2x$, the feasible region is given below: 5



10

Points	z
$P_5 \equiv (2,4)$	52,000
$P_3 \equiv (5,1)$	58,000
$P_4 \equiv (6.5,0)$	65,000

15

Minimum $z = 52{,}000$ at $P_5 \equiv (2,4)$.

The increase of total cost $= 52{,}000 - 50{,}000 = Rs.\,2000$. 10

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12. (a) Let $\mathbf{A} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$.

Find x and y such that $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}_3$, where $\mathbf{I}_3$ is the identity matrix of order 3 and $\mathbf{A}^{\mathrm{T}}$ represents the transpose of $\mathbf{A}$.

(b) Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find constants p and q such that $\mathbf{A}^3 + p\mathbf{A} = q\mathbf{I}_3$, where $\mathbf{I}_3$ is the identity matrix of order 3.

Deduce that there is a square matrix $\mathbf{B}$ of order 3 such that $\mathbf{BA} = \mathbf{I}_3$.

Consider the following system of linear equations:

$$\begin{aligned} y + z &= 1 \\ x + z &= 2 \\ x + y &= 5 \end{aligned}$$

By taking $\mathbf{H} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, show that the matrix equation $\mathbf{AX} = \mathbf{H}$ represents the above system of linear equations.

Hence solve the above system of linear equations.

(a) $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix} \Rightarrow A^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{pmatrix}$. 5

$$AA^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix}. \quad 5$$

Thus, $AA^T = I \Rightarrow \frac{1}{9}\begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 5

$$\Rightarrow \begin{cases} x+4+2y = 0 \\ 2x+2-2y = 0 \\ x^2+4+y^2 = 9 \end{cases} \Rightarrow \begin{cases} x+2y = -4 \\ x-y = -1 \\ x^2+y^2 = 5 \end{cases} \text{ --- (1)} \quad 15$$

Solving equations in (1), we get $x = -2$ and $y = -1$. 10

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(b) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$A^2 = AA = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and 5

$A^3 = A^2A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$. 5

Thus, $A^3 + pA = qI_3 \Rightarrow \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} + p\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 5

$\Rightarrow \begin{pmatrix} 2 & 3+p & 3+p \\ 3+p & 2 & 3+p \\ 3+p & 3+p & 2 \end{pmatrix} = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$ --- (2) 5

Comparing corresponding elements of matrix equation (2), we get $q = 2$ and 5 $p = -3$. 5

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Now we have $A^3 - 3A = 2I_3 \Rightarrow \left[\frac{1}{2}(A^2 - 3I_3)\right]A = I_3$. --- (3) 10

Let $B = \frac{1}{2}(A^2 - 3I_3)$.

Then by (3) we have $BA = I_3$. 5

So there is a square matrix B of order 3 such that $BA = I_3$, where 10

$$B = \frac{1}{2}(A^2 - 3I_3) = \frac{1}{2}\left\{\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

25

$AX = H \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$. 10

Equating corresponding elements of the matrix equation, we get

$$\begin{cases} \quad y + z = 1 \\ x \quad + z = 2. \\ x + y \quad = 5 \end{cases}$$ 5

Thus, the matrix equation $AX = H$ represents the above system of linear equations. 5

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5

$AX = H \Rightarrow B(AX) = BH \Rightarrow (BA)X = BH \Rightarrow I_3X = BH \Rightarrow X = BH$. 10

Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = BH = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

Thus, $x = 3, y = 2$ and $z = -1$. 15 5

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13.(*a*) Two unbiased standard six-sided dice I and II with faces marked 1, 2, 3, 4, 5, 6 are tossed. Let x and y be the numbers landed in die I and die II respectively.

Let A and B be the events defined by

$A : x \leq y$, and

$B : x + y$ is an odd integer.

Find $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \mid B)$.

(*b*) (*i*) Find the number of different permutations that can be formed from the ten letters of the word "STATISTICS".

(*ii*) Find the number of different combinations that can be formed from four letters taken from the ten letters of the word "STATISTICS".

(a)

	1	2	3	4	5	6	
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	A
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	

$$A = \left\{\begin{matrix}(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\end{matrix}\right\}$$ 10

	1	2	3	4	5	6	
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	B
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	

$$B = \left\{\begin{matrix}(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), \\ (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\end{matrix}\right\}$$ 10

$$A \cap B = \{(1,2), (1,4), (1,6), (2,3), (2,5), (3,4), (3,6), (4,5), (5,6)\}$$ 10

Notice that each pair of numbers is equally likely, Therefore, we find

$$P(A) = \frac{\text{Number of pairs in } A}{\text{Total number of all possible pairs in the sample space}} = \frac{21}{36} = \frac{7}{12}. \quad 10$$

$$P(B) = \frac{\text{Number of pairs in } B}{\text{Total number of all possible pairs in the sample space}} = \frac{18}{36} = \frac{1}{2}. \quad 10$$

$$P(A \cap B) = \frac{\text{Number of pairs in } A \cap B}{\text{total number of all possible pairs in the sample space}} = \frac{9}{36} = \frac{1}{4}.$$

10

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}. \quad 10$$

70

Aliter

Using total probability law, we find,

$$P(A) = \sum_{y=1}^{y=6} P(A|\{y\})\, P(\{y\}) \quad 10$$

$$= \frac{1}{6}\left(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}\right)$$

$$= \frac{1}{6} \times \frac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{6}{2} \times 7$$

$$= \frac{7}{12}. \quad 10$$

Using total probability law, we find,

$$P(B) = P(x + y \text{ is odd} \mid \text{y is odd}) P(y \text{ is odd}) + P(x + y \text{ is odd} \mid \text{y is even})\, P(y \text{ is even})$$

$$= P(x \text{ is even}) \times P(y \text{ is odd}) + P(x \text{ is odd}) \times P(y \text{ is even}) \quad 10$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{2}. \quad 10$$

Using total probability law, we find,

$P(A \cap B) = \sum_{y=1}^{y=6} P(A \cap B | \{y\}) \times P(\{y\})$ 10

$= P(x \leq y \text{ and } x + y \text{ is odd } |\{y\}) \times P(\{y\})$

$= \frac{1}{6} \times (0 + \frac{1}{6} + \frac{1}{6} + \frac{2}{6} + \frac{2}{6} + \frac{3}{6})$.

$= \frac{1}{6} \times \frac{9}{6}$

$= \frac{1}{4}$. 10

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$. 10

$\boxed{70}$

(b) (i) In the term STATISTICS, the numbers of occurrences of each of the letters are:

$S - 3;\ T - 3;\ A - 1;\ I - 2;\ C - 1$ 10

Number of different permutations of the ten letters $= \frac{10!}{3!3!2!}$ 10

$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{3 \times 2 \times 2}$

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 2$

$= 50400.$ 10

$\boxed{30}$

(ii) Number of combinations with all four letters different from each other $= 5_{C_4} = 5$ 5

5

Number of combinations with two pairs with two equal letters in each pair

5 $= 3_{C_2} = 3$ 5

5 Number of combinations with two equal letters and two different letters

$= 3_{C_1} \times 4_{C_2} = 18$ 5

5 Number of combinations with three equal letters and a different another letter

$= 2_{C_1} \times 4_{C_3} = 8$ 5

∴ The number of different combinations from four letters in the term STATISTICS $= 5 + 3 + 18 + 8 = 34.$ 10

$\boxed{50}$

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14. Fruits are packed in three boxes, A, B and C, so that box A contains only 7 mangoes, box B contains 4 mangoes and 3 pears and box C contains 5 apples and 2 pears. Suppose that a box is selected at random and 2 fruits are randomly picked one after the other without replacement from the selected box.
Assuming that the selection of each box is equally likely, find the probability that
(i) both fruits selected are mangoes,
(ii) at least one of the selected fruits is a mango,
(iii) both fruits selected are mangoes given that one is a mango,
(iv) fruits are of different kinds.

Box A: 7 mangoes
Box B: 4 mangoes and 3 pears
Box C: 5 apples and 2 pears

A box is at random and 2 fruits are randomly picked without replacement from the selected box. Each box is selected with equal probability.
Thus, $P(A)=P(\text{Box B})=P(\text{Box C}) = \frac{1}{3}$ 5

(i) $P(\text{Both are mangoes}) = P(\text{Both are mangoes}|\text{Box A})\times P(\text{Box A})+$

$P(\text{both are mangoes}|\text{Box B})\times P(\text{Box B})+$ 10

5 $P(\text{Both are mangoes}|\text{Box C})\times P(\text{Box C})$

$= 1\times\frac{1}{3}+\frac{{}^4C_2}{{}^7C_2}\times\frac{1}{3}+0\times\frac{1}{3}$ 5

$=\frac{1}{3}\left(1+\frac{2}{7}\right)$ 5

$=\frac{3}{7}.$ 5

35

$P(\text{At least one of the selected fruits is a mango}) = 1 - P(\text{None of the fruits is a mango})$

$=1-\{P(\text{None is a mango}|A)\times P(\text{Box A})+$ 10

$P(\text{None is a mango}|B)\times P(\text{Box B})+$ 10

$P(\text{None is a mango}|C)\times P(\text{Box C})\}$

$=1-\left(0+\frac{{}^3C_2}{{}^7C_2}\times\frac{1}{3}+1\times\frac{1}{3}\right)$

5 5

$=1-\left(\frac{3\times2}{7\times6}\times\frac{1}{3}+1\times\frac{1}{3}\right)$ 5

$=1-\left(\frac{8}{7}\right)\times\frac{1}{3}$

$=\frac{13}{21}$. 5

40

(iii) P (Both are mangoes | One is a mango) $=\dfrac{\text{P(Both are mangoes)}}{\text{P(One is a mango)}}$ 15

$=\dfrac{3/7}{13/21}$ 5 5

$=\frac{9}{13}$. 5

35

(iv) P(Fruits are of different kinds) = P(Fruits are of different kinds|Box A)×P(Box A) + 15

P(Fruits are of different kinds|Box B)×P(Box B) +

P(Fruits are of different kinds|Box C)×P(Box C)

$=0\times\frac{1}{3}+\frac{{}^4C_1\times{}^3C_1}{{}^7C_2}\times\frac{1}{3}+\frac{{}^5C_1\times{}^2C_1}{{}^7C_2}\times\frac{1}{3}$

5 5 5

$=\left(\frac{4\times3\times2}{7\times6}\right)\times\frac{1}{3}+\left(\frac{5\times2\times2}{7\times6}\right)\times\frac{1}{3}$ 5

$=\left(\frac{22}{21}\right)\times\frac{1}{3}$

$=\frac{22}{63}$. 5

40

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15. A continuous random variable X has an exponential distribution with probability density function $f(x)$ given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0, \\ 0 & , \text{otherwise,} \end{cases}$$

where $\lambda (> 0)$ is a parameter.

Find the mean and the variance of X.

The lifetime X of an electric equipment is exponentially distributed with a mean of 2 years. Find the cumulative distribution function of X and hence find the median of X.
(You may take $e^{-0.7} \simeq 0.5$.)

An equipment is randomly selected. Find the probability that

(i) the life time of the equipment will exceed $1\frac{1}{2}$ years,

(ii) the equipment will fail before 2 years, given that the equipment had lasted more than $1\frac{1}{2}$ years.
(You need not simplify the answers.)

The random variable X has an exponential distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

where, λ (>0) is a parameter.

Mean of $X = E(X) = \int_0^{\infty} x \, \lambda e^{-\lambda x} \, dx$ 10

$= \int_0^{\infty} x \, \frac{d(-e^{-\lambda x})}{dx} dx$ 5

$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} \, dx$ 5

$= \frac{e^{-\lambda x}}{-\lambda}\Big|_0^{\infty}$ 5

$= \frac{1}{\lambda}.$ 5

30

Variance of $X = V(X) = E(X^2) - (E(X))^2.$ 10

$E(X^2) = \int_0^{\infty} x^2 \, \lambda e^{-\lambda x} \, dx$ 5

$= \int_0^{\infty} x^2 \, \frac{d(-e^{-\lambda x})}{dx} dx$ 5

$$= x^2 e^{-\lambda x}\Big|_{\infty}^{0} + \frac{2}{\lambda}\int_0^{\infty} x\, e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda^2}. \quad 5$$

$$V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}. \quad 5$$

$\boxed{30}$

Let X denote the lifetime of an electric equipment.

Given that X has an exponential distribution with a mean life time of 2 years.

Thus, density function of X is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where, $\frac{1}{\lambda} = 2$ and $\lambda = 0.5$. 5

Cumulative distribution function of X, $F(X)$ is given by:

$$F(X) = \underbrace{\int_0^x f(x)\, dx}_{10} = \int_0^x \lambda e^{-\lambda x} dx, \text{ where } \lambda=0.5. \quad 5$$

$$= e^{-\lambda x}\Big|_x^0, \text{ where } \lambda=0.5. \quad 5$$

$$= 1 - e^{-\lambda x}, \text{ where } \lambda=0.5.$$

$$= 1 - e^{-0.5x}. \quad 5$$

$\boxed{30}$

Median of X can be found as the value of x that corresponds to $F(x) = 0.5$. 10

This means $1 - e^{-0.5x} = 0.5$ or equivalently $e^{-0.5x} = 0.5$. 10

Given that $e^{-0.7}$=0.5. Thus, we find $0.5x = 0.7$. 5

Thus, median of X is 1.4. 5

$\boxed{30}$

(i) $P(X > 1.5) = 1 - P(X \leq 1.5) = 1 - (1 - e^{-0.75})$

5 5

The probability that the lifetime exceeds 1.5 years = $e^{-0.75}$. 5

$\boxed{15}$

(ii) Need $P(X < 2|X > 1.5)$.

$$P(X < 2|X > 1.5) = \frac{P(1.5 < X < 2)}{P(X > 1.5)} \quad 5$$

Notice that $P(1.5 < X < 2) = P(X < 2) - P(X < 1.5)$ 5

$$= F(2) - F(1.5)$$

$$= (1 - e^{-1}) - (1 - e^{-0.75}) \quad 5$$

$$= e^{-0.75} - e^{-1}.$$

20

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16. The mean and the standard deviation of the set of values $\{x_i : i = 1, 2, \ldots, n\}$ are μ and σ respectively. Find the mean and the standard deviation of the set of values $\{ax_i + b : i = 1, 2, \ldots, n\}$, where a and b are constants.

The following table summarises the ages (recorded to the nearest year) at the initial diagnosis of high blood sugar of a group of 70 diabetic patients.

Age	Number of patients
10 – 20	9
20 – 30	12
30 – 40	32
40 – 50	14
50 – 60	3

(i) Using a suitable linear transformation or otherwise, calculate the mean and the standard deviation of the given frequency distribution.

(ii) Find the inter-quartile range of the above distribution.

(iii) Two more patients who were both initially diagnosed with high blood sugar at the age of 55 joined the group. Find the inter-quartile range of the frequency distribution of the initial age of diagnosis of high blood sugar of all 72 patients.

Let μ and σ be the mean and standard deviation of the set of values $\{x_i : i = 1,2,\cdots,n\}$ respectively.

Mean of the y_i values $= \mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i$ 5

$= \frac{1}{n}\sum_{i=1}^{n}(ax_i + b)$

$= a\frac{1}{n}\sum_{i=1}^{n}(x_i) + b$

$= a\mu + b.$ 5

10

Standard deviation of the y_i values $\sigma_y = \sqrt{\sigma^2}$, where, ${\sigma_y}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu_y)^2.$ 5

Thus, $\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}(ax_i + b - a\mu - b)^2.$ 5

$= a^2\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$

$= a^2\sigma^2$ 5

Therefore, $\sigma_y = a\sigma.$ 5

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Let $y_i = (x_i - 35)/10$, where x_i denote the age.

Class interval for y_i	No. of patients f_i	Class mid point (m_i)	$f_i m_i$	$f_i m_i^2$
$-2.5 \rightarrow -1.5$	9	-2	-18	36
$-1.5 \rightarrow -0.5$	12	-1	-12	12
$-0.5 \rightarrow 0.5$	32	0	0	0
$0.5 \rightarrow 1.5$	14	1	14	14
$1.5 \rightarrow 2.5$	3	2	6	12
Total	70		-10	74

5 5

Mean of the y_i values $= \mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{\sum_{i=1}^{n} f_i m_i}{\sum_{i=1}^{n} f_i}$ 5

$= -\frac{10}{70} = -\frac{1}{7}.$ 5 5

According to previous results, mean of the x_i values $= 35 + 10 \times \mu_y = 35 - 1.43 = 33.57.$

$\boxed{25}$

Standard deviation of the y_i values $= \sigma_y = \sqrt{\sigma_y^2},$

where $\sigma_y^2 \frac{1}{\sum_{i=1}^{n} f_i}\left(\sum_{i=1}^{n} f_i m_i^2 - \frac{(\sum_{i=1}^{n} f_i m_i)^2}{\sum_{i=1}^{n} f_i}\right).$

$\Rightarrow \sigma_y^2 = \frac{1}{70}(74 - 100/70) = \frac{1}{70^2} \times 5080 = \frac{50.8 \times 10^2}{70^2}.$ 10

Thus, $\sigma_y = \frac{7.1 \times 10}{70} = 1.02.$ 5

Standard deviation of the 5 lues = 5 .02 = 10.2. 5

$\boxed{30}$

(ii) $\frac{n}{4} = \frac{70}{4} = 17.5$ 5

First quartile belongs to the 2nd class interval.

First quartile = $Q_1 = 20 + \frac{10}{12} \times (17.5 - 9) = 20 + \frac{5}{6} \times 8.5 = 20 + \frac{42.5}{6} = 20 + 7.1$ 5

$= 27.1$ 5

$\frac{3n}{4} = \frac{210}{4} = 52.5$ 5

Third quartile belongs to the 3rd class interval.

Third quartile = $Q_3 = 30 + \frac{10}{32} \times (52.5 - 21) = 30 + \frac{5}{16} \times 31.5$

5

5

$$= 30 + \frac{157.5}{16}$$
$$= 30 + 9.8$$
$$= 39.8.$$

Inter-quartile range = $Q_3 - Q_1 = 39.8 - 27.1 = 12.7$ 5 5

35

(iii) The ages of the persons newly added to the group belong to the last class interval.

After these two observations are added, $\frac{n}{4} = \frac{72}{4} = 18.$ 5

New first quartile belongs to the 2nd class interval.

New first quartile = $Q_1 = 20 + \frac{10}{12} \times (18 - 9) = 20 + \frac{15}{2} = 20 + 7.5 = 27.5.$ 5 5

After addition of the two observations, $\frac{3n}{4} = \frac{216}{4} = 54.$ 5

New third quartile belongs to the 4th class interval.

Third quartile = $Q_3 = 40 + \frac{10}{14} \times (54 - 53) = 40 + \frac{5}{7} = \frac{285}{7} = 40.7$ 5 5

New inter-quartile range = $40.7 - 27.5 = 13.2.$ 5

35

17. Duration of activities in a project and the flow of activities are described in the following table:

Activity	Immediate predecessor(s)	Duration (in months)
A	–	2
B	A	2
C	A	3
D	B, C	4
E	B, D	5
F	–	8
G	E, F	1
H	E, G	2
I	H	4

(i) Construct the project network.

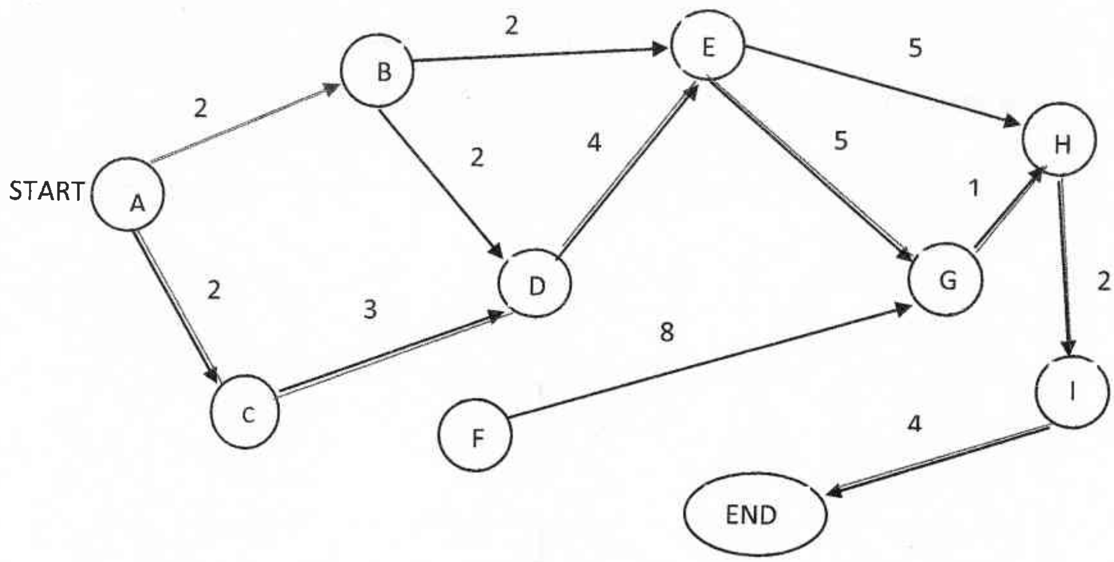
(ii) Prepare an activity schedule that includes earliest start time, earliest finish time, latest start time, latest finish time and float for each activity.

(iii) What are the activities that cannot be delayed without extending the total duration of the project?

(iv) Find the total duration of the project.

(v) Due to external reasons, activity F is expected to take one more month than the regular duration. Determine whether the project can still be completed within the total duration calculated in part (iv) above.

(i)



40

(ii)

Activity	Earliest Start time	Earliest Finish time	Latest Start time	Latest Finish time	Float
A	0	2	0	2	0
B	2	4	3	5	1
C	2	5	2	5	0
D	5	9	5	9	0
E	9	14	9	14	0
F	0	8	6	14	6
G	14	15	14	15	0
H	15	17	15	17	0
I	17	21	17	21	0

10 for each column

50

(iii) Critical path is the line joining A, C, D, E, G, H, I in that order. 10

Therefore, activities that cannot be delayed without delaying the project completion time are: 10

$A, C, D, E, G, H, I.$

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(iv) Total duration of the project $= (2 + 3 + 4 + 5 + 1 + 2 + 4) = 21$ months. 20

20

(v) Activity F has a floating time of 6 months. 10

Hence, even if the activity F takes one more month than anticipated, the project can still be completed within the total duration calculated in part (iv). 10

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