Confidential

NEW

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2019

10 - Combined Mathematics I

NEW Syllabus

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.

3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details. For the subject 51 Art, marks for Papers 01, 02 and 03 should be entered numerically in the mark sheets.

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1. Using the Principle of Mathematical Induction, prove that $\sum_{r=1}^{n}(2r-1) = n^2$ **for all** $n \in \mathbb{Z}^+$.

For $n = 1$, L.H.S. $= 2 \times 1 - 1 = 1$ and R.H.S. $= 1^2 = 1$ 5

$\therefore$ The result is true for $n = 1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} (2r-1) = p^2$. 5

Now $\sum_{r=1}^{p+1} (2r-1) = \sum_{r=1}^{p} (2r-1) + \big(2(p+1) - 1\big)$ 5

$= p^2 + (2p + 1)$

$= (p+1)^2$. 5

Hence, if the result is true for $n = p$, then it is true for $n = p + 1$. We have already proved that the result is true for $n = 1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. 5

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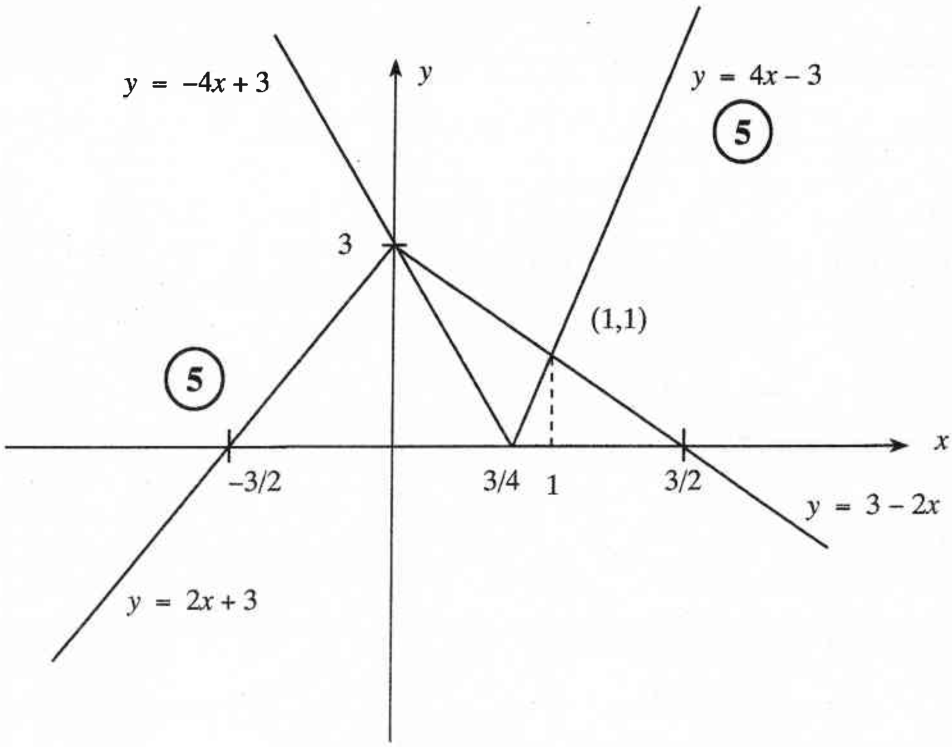
2. **Sketch the graphs of $y=|4x-3|$ and $y=3-2|x|$ in the same diagram.**
Hence or otherwise, find all real values of x satisfying the inequality $|2x-3|+|x|<3$.



At the point of intersections of the graphs

$$4x - 3 = 3 - 2x \Rightarrow x = 1 \quad 5$$
$$-4x + 3 = 3 + 2x \Rightarrow x = 0$$

From the graphs, we have,

$$|4x-3| < 3 - 2|x| \quad \Leftrightarrow \quad 0 < x < 1$$

$$\therefore \ |4x-3| + |2x| < 3 \quad \Leftrightarrow \quad 0 < x < 1$$

Replacing x by $\frac{x}{2}$, we get

$$|2x-3| + |x| < 3 \quad \Leftrightarrow \quad 0 < x < 2. \quad 5$$

Hence, the set of all values of x satisfying

$|2x-3| + |x| < 3$ is $\{x : 0 < x < 2\}$. 5

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Aliter

For the graphs 5 + 5 , as before.

Aliter for values of x

$|2x-3| + |x| < 3$

Case (i) $x \leq 0$:

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3-x < 3$

$\Leftrightarrow 3x > 0$

$\Leftrightarrow x > 0$

Hence, in this case, no solutions exist.

Case (ii) $0 < x \leq \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3+x < 3$

$\Leftrightarrow x > 0$

Hence, in this case, the solutions are the values of x satisfying $0 < x \leq \frac{3}{2}$.

Case (iii) $x > \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow 2x-3+x < 3$

$\Leftrightarrow 3x < 6$

$\Leftrightarrow x < 2$

Hence, in this case, the solutions are the values of x satisfying $\frac{3}{2} < x < 2$.

All 3 cases with correct solutions	10
Any 2 cases with correct solutions	5

Hence, over all, the solutions are values of x satisfying $0 < x < 2$. 5

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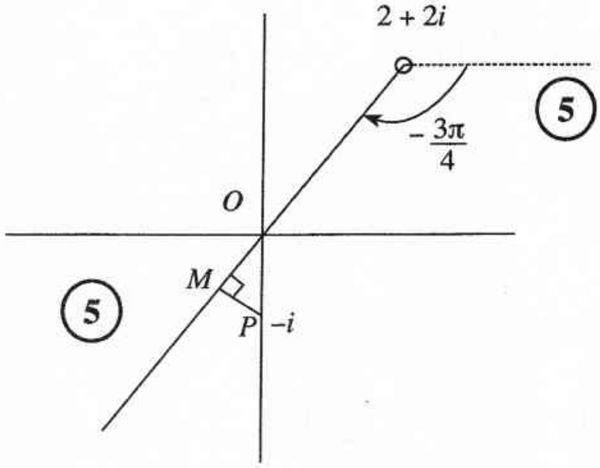
3. **Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\mathrm{Arg}(z-2-2i) = -\frac{3\pi}{4}$.**
Hence or otherwise, find the minimum value of $|i\bar{z}+1|$ such that $\mathrm{Arg}(z-2-2i) = -\frac{3\pi}{4}$.



Note that

$$|i\bar{z}+1| = |i(\bar{z}-i)| = |\bar{z}-i| = |\overline{z+i}|$$
$$= |z+i|$$
$$= |z-(-i)| \quad 5$$

Hence, the minimum of $|i\bar{z}+1|$ is equal to PM. 5

Now, $\mathrm{PM} = 1 \cdot \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. 5

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4. Show that the coefficient of x^6 in the binomial expansion of $\left(x^3 + \frac{1}{x^2}\right)^7$ is 35.

Show also that there does not exist a term independent of x in the above binomial expansion.

$$\left(x^3 + \frac{1}{x^2}\right)^7 = \sum_{r=0}^{7} {}^7C_r \,(x^3)^r \left(\frac{1}{x^2}\right)^{7-r} \quad 5$$

$$= \sum_{r=0}^{7} {}^7C_r \, x^{5r-14}$$

x^6 : $5r - 14 = 6 \Leftrightarrow r = 4.$ 5

$\therefore$ The coefficient of $x^6 = {}^7C_4 = 35$ 5

For the above expansion to have a term independent of x, we must have

$5r - 14 = 0.$ 5

This is not possible as $r \in \mathbb{Z}^+$. 5

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5. Show that $\lim_{x \to 3} \dfrac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \dfrac{1}{2\pi}$.

$$\lim_{x \to 3} \frac{\sqrt{x-2}\ -1}{\sin(\pi(x-3))} = \lim_{x \to 3} \frac{\sqrt{x-2}\ -1}{\sin(\pi(x-3))} \cdot \frac{(\sqrt{x-2}+1)}{(\sqrt{x-2}+1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{x-3}{\sin(\pi(x-3))} \cdot \lim_{x \to 3} \frac{1}{(\sqrt{x-2}+1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{1}{\dfrac{\sin(\pi(x-3))}{\pi(x-3)}} \cdot \frac{1}{\pi} \cdot \frac{1}{2} \quad 5$$

$$= 1 \cdot \frac{1}{\pi} \cdot \frac{1}{2}$$

5

$$= \frac{1}{2\pi} \quad 5$$

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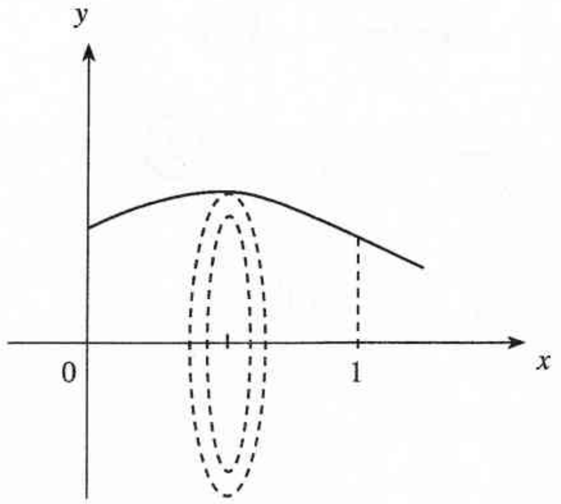
6. The region enclosed by the curves $y=\sqrt{\frac{x+1}{x^2+1}}$, $x=0$, $x=1$ and $y=0$ is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\frac{\pi}{4}(\pi+\ln 4)$.



The volume generated $= \int_0^1 \pi\left(\sqrt{\frac{x+1}{x^2+1}}\right)^2 dx$ 5

$= \pi\left(\int_0^1 \frac{x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx\right)$ 5

$= \pi\left(\frac{1}{2}\ln(x^2+1)\Big|_0^1 + \tan^{-1}x\Big|_0^1\right)$ 5 + 5

$= \pi\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right)$

$= \frac{\pi}{4}\left(\ln 4 + \pi\right)$ 5

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7. Let C be the parabola parametrically given by $x = at^2$ and $y = 2at$ for $t \in \mathbb{R}$, where $a \neq 0$. Show that the equation of the normal line to the parabola C at the point $(at^2, 2at)$ is given by $y + tx = 2at + at^3$.

The normal line at the point $P \equiv (4a, 4a)$ on the parabola C meets this parabola again at a point $Q \equiv (aT^2, 2aT)$. Show that $T = -3$.

$x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2a \cdot \frac{1}{2at} = \frac{1}{t}$ for $t \neq 0$. 5

$\therefore$ The slope of the normal line $= -t$

The equation of the normal at $(at^2, 2at)$ is

$y - 2at = -t\,(x - at^2)$

$y + tx = 2at + at^3$ 5 (This is valid for $t = 0$ also.)

$P \equiv (4a, 4a)$ on $C \Rightarrow t = 2$.

The normal line at P : $y + 2x = 4a + 8a = 12a$ 5

Since it meets C at $(aT^2, 2aT)$, we have

$2aT + 2aT^2 = 12a$. 5

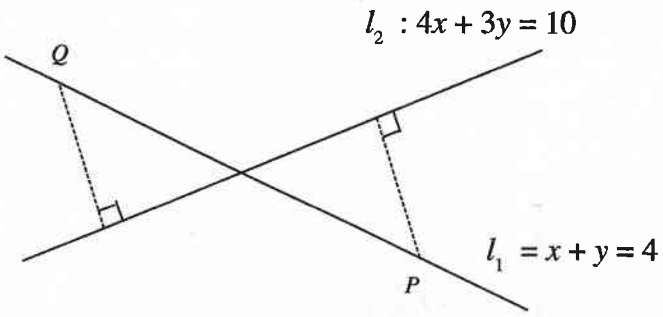
$\Leftrightarrow T^2 + T - 6 = 0 \Leftrightarrow (T-2)(T+3) = 0$

$\Leftrightarrow T = 2$ or $T = -3$

$\therefore T = -3$ 5

$\boxed{25}$

8. Let l_1 and l_2 be the straight lines given by $x+y=4$ and $4x+3y=10$, respectively. Two distinct points P and Q are on the line l_1 such that the perpendicular distance from each of these points to the line l_2 is 1 unit. Find the coordinates of P and Q.



Any point on the line l_1 can be written in the form

$(t, 4-t),\ t \in \mathbb{R}$. 5

Let $P \equiv (t_1, 4-t_1)$

Perpendicular distance from P to l_2 $= \dfrac{\left|4t_1 + 3(4-t_1) - 10\right|}{\sqrt{4^2+3^2}} = 1$ 5

$\therefore\ |t_1 + 2| = 5$

$\therefore\ t_1 = -7$ or $t_1 = 3$ 5

The coordinates of P and Q are

$(-7, 11)$ and $(3, 1)$. 5 + 5

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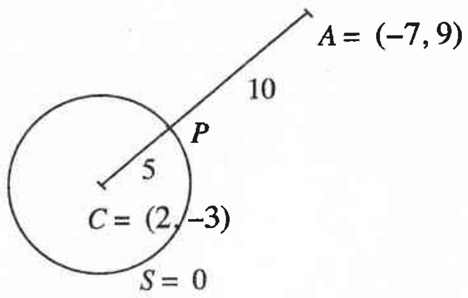
9. Show that the point $A \equiv (-7, 9)$ lies outside the circle $S \equiv x^2+y^2-4x+6y-12=0$. Find the coordinates of the point on the circle $S=0$ nearest to the point A.

The centre C of $S = 0$ is $(2, -3)$. 5

The radius R of $S = 0$ is $\sqrt{4+9+12} = \sqrt{25} = 5$. 5

$CA^2 = 9^2 + 12^2 = 15^2 \Rightarrow CA = 15 > R = 5$. 5

$\therefore$ Point A lies outside the given circle.



The point on the circle $S = 0$ nearest to point A is the point P at which CA meets $S = 0$.

Note that $CP : PA = 5 : 10$

$= 1 : 2$ 5

$$\therefore\ P \equiv \left(\frac{2 \times 2 + 1(-7)}{3},\ \frac{2(-3) + 1 \times 9}{3}\right)$$

i.e. $P \equiv (-1,\ 1)$ 5

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10. Let $t = \tan\frac{\theta}{2}$ for $\theta \neq (2n+1)\pi$, where $n \in \mathbb{Z}$. Show that $\cos\theta = \frac{1-t^2}{1+t^2}$.

Deduce that $\tan\frac{\pi}{12} = 2-\sqrt{3}$.

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad 5$$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} \quad 5 \quad \text{for } \theta \neq (2n+1)\pi.$$

$$= \frac{1-t^2}{1+t^2}$$

Let $\theta = \frac{\pi}{6}$. Then $\frac{\sqrt{3}}{2} = \frac{1-t^2}{1+t^2}$

5

$$\Rightarrow \sqrt{3}\,(1+t^2) = 2\,(1-t^2)$$

$$(2+\sqrt{3})\,t^2 = 2-\sqrt{3}$$

$$\therefore \quad t^2 = \frac{(2-\sqrt{3})}{(2+\sqrt{3})} \quad 5$$

$$= (2-\sqrt{3})^2$$

$$\Rightarrow t = \tan\frac{\pi}{12} = 2-\sqrt{3} \quad 5 \quad \left(\because \tan\frac{\pi}{12} > 0\right)$$

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11. (*a*) Let $p \in \mathbb{R}$ and $0 < p \leq 1$. Show that 1 is **not** a root of the equation $p^2x^2 + 2x + p = 0$.
Let α and β be the roots of this equation. Show that α and β are both real.
Write down $\alpha + \beta$ and $\alpha\beta$ in terms of p, and show that
$$\frac{1}{(\alpha-1)} \cdot \frac{1}{(\beta-1)} = \frac{p^2}{p^2+p+2}.$$
Show also that the quadratic equation whose roots are $\frac{\alpha}{\alpha-1}$ and $\frac{\beta}{\beta-1}$ is given by $(p^2+p+2)x^2 - 2(p+1)x + p = 0$ and that both of these roots are positive.

(*b*) Let c and d be two **non-zero** real numbers and let $f(x) = x^3 + 2x^2 - dx + cd$. It is given that $(x-c)$ is a factor of $f(x)$ and that the remainder when $f(x)$ is divided by $(x-d)$ is cd. Find the values of c and d.
For these values of c and d, find the remainder when $f(x)$ is divided by $(x+2)^2$.

(*a*) Suppose that 1 is a root of $p^2x^2 + 2x + p = 0$.

By substituting $x = 1$, we must have $p^2 + 2 + p = 0$. 5

This is impossible, as $p > 0$ implies that $p^2 + 2 + p > 0$. 5

$\therefore$ 1 is not a root of $p^2x^2 + 2x + p = 0$ **10**

The discriminant $\Delta = 2^2 - 4p^2 \cdot p$ 10

$= 4(1 - p^3)$

$\geq 0 \ (\because 0 < p \leq 1)$ 5

$\therefore$ α and β are both real. 5 **20**

$\alpha + \beta = -\frac{2}{p^2}$ and $\alpha\beta = \frac{1}{p}$ 5 + 5

Now,

$$\frac{1}{(\alpha-1)} \cdot \frac{1}{(\beta-1)} = \frac{1}{(\alpha\beta - (\alpha+\beta) + 1)} \quad 5$$

$$= \frac{1}{\frac{1}{p} + \frac{2}{p^2} + 1}$$

$$= \frac{p^2}{p^2 + p + 2}. \quad 5$$

20

Now

$$\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1}=\frac{\alpha(\beta-1)+\beta(\alpha-1)}{(\alpha-1)(\beta-1)}$$

$$=\frac{2\alpha\beta-(\alpha+\beta)}{(\alpha-1)(\beta-1)} \quad 5$$

$$=\left(\frac{2}{p}+\frac{2}{p^2}\right)\cdot\frac{p^2}{p^2+p+2} \quad 5$$

$$=\frac{2(p+1)}{p^2}\cdot\frac{p^2}{p^2+p+2}$$

$$=\frac{2(p+1)}{p^2+p+2} \quad 5$$

and

$$\frac{\alpha}{\alpha-1}\cdot\frac{\beta}{\beta-1}=\frac{\alpha\beta}{(\alpha-1)(\beta-1)}$$

$$=\frac{1}{p}\cdot\frac{p^2}{p^2+p+2}$$

$$=\frac{p}{p^2+p+2}. \quad 5$$

Hence, the required quadratic equation is given by

$$x^2-\frac{2(p+1)}{p^2+p+2}x+\frac{p}{p^2+p+2}=0 \quad 10$$

$$\Rightarrow \quad (p^2+p+2)x^2-2(p+1)x+p=0 \quad 5$$

35

Moreover, note that $\frac{\alpha}{(\alpha-1)}$ and $\frac{\beta}{(\beta-1)}$ are both real,

$$\frac{\alpha}{(\alpha-1)}+\frac{\beta}{(\beta-1)}=\frac{2(p+1)}{p^2+p+2}>0, \ (\because\ p>0),$$

5

and $\frac{\alpha}{(\alpha-1)}\cdot\frac{\beta}{(\beta-1)}=\frac{p}{p^2+p+2}>0, \ (\because\ p>0).$

Hence, both of these roots are possitive. 5

10

(b) $f(x) = x^3 + 2x^2 - dx + cd$

Since $(x - c)$ is a factor, $f(c) = 0$. 5

$\Rightarrow c^3 + 2c^2 - dc + cd = 0$ 5

$\Rightarrow c^2 (c + 2) = 0$

$\Rightarrow c = -2 \quad (\because c \neq 0)$ 5

Since, when $f(x)$ is divided by $(x - d)$, the remainder is cd, we have

$$f(d) = cd.$$ 5

$\Rightarrow d^3 + 2d^2 - d^2 + cd = cd$ 5

$\Rightarrow d^3 + d^2 = 0$

$\Rightarrow d^2 (d + 1) = 0$

$\Rightarrow d = -1 \quad (\because d \neq 0)$ 5

$\therefore \quad c = -2$ and $d = -1$. $\boxed{30}$

$f(x) = x^3 + 2x^2 + x + 2.$

Let $Ax + B$ be the remainder, when $f(x)$ is divided by $(x + 2)^2$.

Then $f(x) \equiv (x + 2)^2 Q(x) + (Ax + B)$, where $Q(x)$ is a polynomial of degree 1.

So, $x^3 + 2x^2 + x + 2 \equiv (x + 2)^2 Q(x) + Ax + B.$ 5

Substituting $x = -2$, we obtain $0 = -2A + B.$ 5

By differentiating, we have

$$3x^2 + 4x + 1 = (x + 2)^2 Q'(x) + 2Q(x)(x + 2) + A.$$ 5

Again by substituting $x = -2$, we obtain

$$12 - 8 + 1 = A$$ 5

$$\therefore \; A = 5 \text{ and } B = 10$$

Hence the remainder is $5x + 10$. 5 $\boxed{25}$

Hence, the number of different passwords that can be formed by choosing 3 elements

from P_1 and the other 3 elements from $P_2 = 28800 + 864000 + 864000 + 28800 = 1785600$

10

$\boxed{50}$

(b) $U_r = \dfrac{1}{r(r+1)(r+3)(r+4)}$ and $V_r = \dfrac{1}{r(r+1)(r+2)}$; $r \in \mathbb{Z}^+$.

Then,

$$V_r - V_{r+2} = \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+2)(r+3)(r+4)} \quad 5$$

$$= \frac{(r+3)(r+4) - r(r+1)}{r(r+1)(r+2)(r+3)(r+4)}$$

$$= \frac{6(r+2)}{r(r+1)(r+2)(r+3)(r+4)} \quad 5$$

$$= 6\,U_r \quad 5$$

$\boxed{15}$

Now note that,

$$\begin{aligned}
r = 1;&\quad 6\,U_1 = V_1 - \cancel{V_3},\\
r = 2;&\quad 6\,U_2 = V_2 - \cancel{V_4},\\
r = 3;&\quad 6\,U_3 = \cancel{V_3} - V_5,\\
r = 4;&\quad 6\,U_4 = \cancel{V_4} - V_6,\\
&\quad\vdots\\
r = n-3;&\quad 6\,U_{n-3} = V_{n-3} - \cancel{V_{n-1}}\\
r = n-2;&\quad 6\,U_{n-2} = V_{n-2} - \cancel{V_n}\\
r = n-1;&\quad 6\,U_{n-1} = \cancel{V_{n-1}} - V_{n+1}\\
r = n;&\quad 6\,U_n = \cancel{V_n} - V_{n+2}
\end{aligned}$$

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10