Confidential

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2018

10 - Combined Mathematics - I

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

G.C.E. (A/L) Examination - 2018
10 - Combined Mathematics

Distribution of Marks

Paper I

Part A :	**10 X 25**	**=**	**250**
Part B :	**05 X 150**	**=**	**750**

Total	**=**	**1000/10**
Paper I Final Mark	**=**	**100**

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{Z}^+$.

For $n=1$, L.H.S. $= 1^3 = 1$ and R.H.S. $= \frac{1}{4}\cdot 1^2(1+1)^2 = 1$. 5

$\therefore$ The result is true for $n=1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} r^3 = \frac{1}{4}p^2(p+1)^2$. 5

Now $\sum_{r=1}^{p+1} r^3 = \sum_{r=1}^{p} r^3 + (p+1)^3$ 5

$$= \frac{1}{4}p^2(p+1)^2 + (p+1)^3$$

$$= (p+1)^2 \frac{[p^2+4p+4]}{4}.$$

$$= \frac{1}{4}(p+1)^2(\overline{p+1}+1)^2.$$ 5

Hence if the result is true for $n = p$, then it is also true for $n = p+1$. We have already proved that the result is true for $n=1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$.

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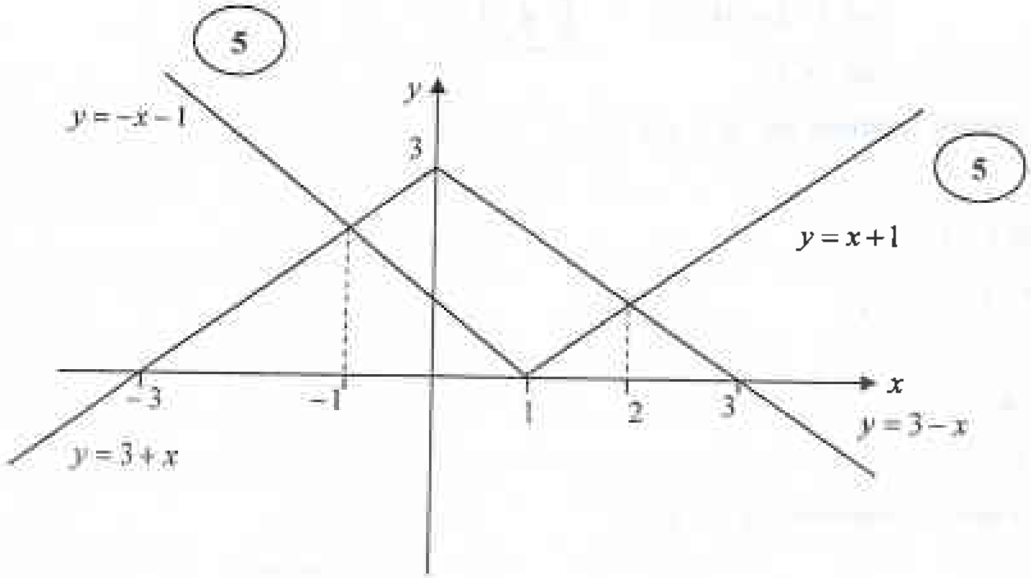
2. Sketch the graphs of $y=3-|x|$ and $y=|x-1|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $|x|+|x-1|\le 3$.



At the points of intersections $-x+1=3+x$ or $x-1=3-x$

i.e. $x=-1$ or $x=2$. 5

Note that $|x|+|x-1|\le 3$

$\Leftrightarrow \quad |x-1|\le 3-|x|$ 5

Hence, from the graph, the solutions are the value of x satisfying $-1\le x\le 2$. 5

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Aliter I

$|x|+|x-1|\le 3$

Case (i) $x\le 0 : |x|+|x-1|\le 3 \Leftrightarrow -x-(x-1)\le 3$

$\Leftrightarrow -2x+1\le 3$ 5

$\Leftrightarrow x\ge -1$

In this case the solutions are $-1\le x\le 0$

Case (ii) $0 < x \leq 1$,

$|x| + |x-1| \leq 3$

$\Leftrightarrow x-(x-1) \leq 3$

$\Leftrightarrow x-(x-1) \leq 3$ 5

$\Leftrightarrow 1 \leq 3$

In this case the solutions are $0 < x \leq 1$.

Case (iii) $1 < x$

$|x| + |x-1| \leq 3$

$\Leftrightarrow x + x - 1 \leq 3$

$\Leftrightarrow 2x \leq 4$

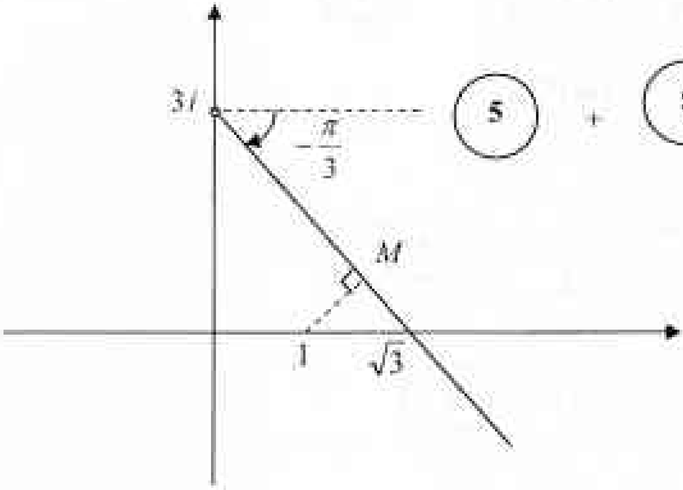
$\Leftrightarrow x \leq 2$

$\therefore$ In this case the solutions are $1 < x \leq 2$.

Hence the solutions are the value of x satisfying $-1 \leq x \leq 2$. 5

3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\mathrm{Arg}(z-3i) = -\frac{\pi}{3}$.

Hence or otherwise, find the minimum value of $|z-1|$ such that $\mathrm{Arg}(\bar{z}+3i) = \frac{\pi}{3}$.



5 + 5

Note that

$$\mathrm{Arg}(\bar{z}+3i) = \frac{\pi}{3}$$

$$\Leftrightarrow \ \mathrm{Arg}(\overline{\bar{z}+3i}) = -\frac{\pi}{3}$$

$$\Leftrightarrow \ \mathrm{Arg}(z-3i) = -\frac{\pi}{3}.$$ 5

Hence the minimum value of $|z-1|$ such that $\mathrm{Arg}(z-3i) = -\frac{\pi}{3}$ is given by NM, 5

where $NM = (\sqrt{3}-1)\sin\frac{\pi}{3} = \frac{(3-\sqrt{3})}{2}$. 5

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4. The coefficients of x and x^4 of the binomial expansion of $\left(x^2+\frac{3k}{x}\right)^8$ are equal. Find the value of the constant k.

$$\left(x^2+\frac{3k}{x}\right)^8=\sum_{r=0}^{8}{}^8C_r(x^2)^r\left(\frac{3k}{x}\right)^{8-r} \quad 5$$

$$=\sum_{r=0}^{8}{}^8C_r(3k)^{8-r}x^{3r-8}$$

$x^1 : 3r-8=1 \Leftrightarrow r=3.$

$x^4 : 3r-8=4 \Leftrightarrow r=4.$ 5

Given ${}^8C_r(3k)^5={}^8C_4(3k)^4$ 5

$$\frac{8!}{3!\,5!}3^5k=\frac{8!}{4!\,4!}3^4 \quad 5$$

$$k=\frac{5}{12}. \quad 5$$

5. Show that $\lim_{x\to 0}\dfrac{1-\cos\left(\frac{\pi x}{4}\right)}{x^2(x+1)} = \dfrac{\pi^2}{32}$.

$$\lim_{x\to 0}\frac{1-\cos\left(\frac{\pi x}{4}\right)}{x^2(x+1)} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{\pi x}{8}\right)}{x^2(x+1)} \quad 5$$

$$= \lim_{x\to 0} 2\left[\frac{\sin\left(\frac{\pi x}{8}\right)}{\left(\frac{\pi x}{8}\right)}\right]^2 \cdot \frac{\pi^2}{64}\cdot\frac{1}{x+1}$$

5

$$= 2\cdot 1\cdot\frac{\pi^2}{64}\cdot\frac{1}{1} \quad 5 \quad 5$$

$$= \frac{\pi^2}{32}. \quad 5$$

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Aliter

$$\lim_{x\to 0}\frac{1-\cos\left(\frac{\pi x}{4}\right)}{x^2(x+1)} = \lim_{x\to 0}\frac{1-\cos\left(\frac{\pi x}{4}\right)}{x^2(x+1)}\cdot\frac{1+\cos\left(\frac{\pi x}{4}\right)}{1+\cos\left(\frac{\pi x}{4}\right)}$$

$$= \lim_{x\to 0}\frac{\sin^2\left(\frac{\pi x}{4}\right)}{x^2(x+1)\left(1+\cos\left(\frac{\pi x}{4}\right)\right)} \quad 5$$

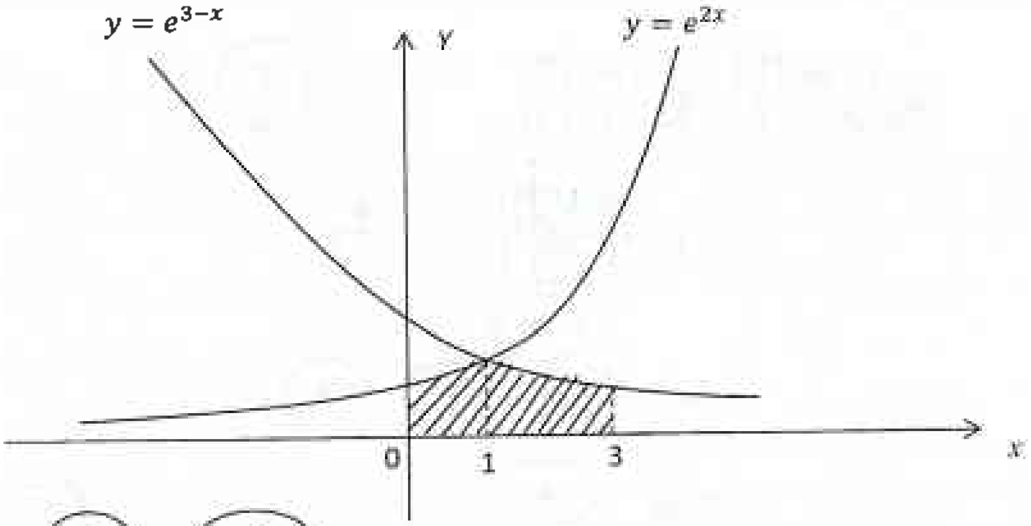
$$= \lim_{x\to 0}\left[\frac{\sin\left(\frac{\pi x}{4}\right)}{\left(\frac{\pi x}{4}\right)}\right]^2\cdot\frac{\pi^2}{16}\cdot\frac{1}{x+1}\cdot\frac{1}{1+\cos\left(\frac{\pi x}{4}\right)}$$

5

$$= 1\cdot\frac{\pi^2}{16}\cdot\frac{1}{1}\cdot\frac{1}{2}$$

$$= \frac{\pi^2}{32}. \quad 5$$

6. Show that the area of the region enclosed by the curves $y = e^{2x}$, $y = e^{3-x}$, $x = 0$, $x = 3$ and $y = 0$ is $\frac{3}{2}\left(e^2 - 1\right)$ square units.



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$$\int_0^1 e^{2x}\,dx + \int_1^3 e^{3-x}\,dx = \frac{e^{2x}}{2}\Big|_0^1 + \frac{e^{3-x}}{(-1)}\Big|_1^3 \quad 5$$

$$= \frac{e^2}{2} - \frac{1}{2} + (-1) + e^2 \quad 5$$

$$= \frac{3e^2}{2} - \frac{3}{2}$$

$$= \frac{3}{2}(e^2 - 1). \quad 5$$

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7. A curve C is given by the parametric equations $x = \ln\left(\tan\frac{t}{2}\right)$ and $y = \sin t$ for $\frac{\pi}{2} < t < \pi$.

Show that $\frac{dy}{dx} = \cos t \sin t$.

Deduce that the gradient of the tangent line drawn to the curve C at the point corresponding to $t = \frac{2\pi}{3}$ is $-\frac{\sqrt{3}}{4}$.

$$x = \ln\left(\tan\frac{t}{2}\right) \qquad y = \sin t$$

$$\frac{dx}{dt} = \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2} \quad 5 \qquad \frac{dy}{dt} = \cos t \quad 5$$

$$= \frac{1}{2\cos\frac{t}{2}\sin\frac{t}{2}} \quad 5$$

$$= \frac{1}{\sin t}$$

Now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \cos t \sin t$ 5

$$\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \cos\frac{2\pi}{3}\sin\frac{2\pi}{3} = -\frac{1}{2} \times \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{4}$$

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10. Show that $\sec^3 x + 2\sec^2 x \tan x + \sec x \tan^2 x = \dfrac{\cos x}{(1-\sin x)^2}$ for $x \neq (2n+1)\dfrac{\pi}{2}$, where $n \in \mathbb{Z}$.

$$\sec^3 a = 2\sec^2 x \tan x + \sec x \tan^2 x$$

$$= \frac{1}{\cos^3 x} + \frac{2\sin x}{\cos^3 x} + \frac{\sin^2 x}{\cos^3 x} \qquad 5$$

$$= \frac{1 + 2\sin x + \sin^2 x}{\cos^3 x}$$

$$= \frac{(1+\sin x)^2}{\cos x(1-\sin^2 x)} \qquad 5$$

$$= \frac{(1+\sin x)^2}{\cos x(1-\sin x)(1+\sin x)} \quad 5 \quad \left(\because x \neq (2n+1)\frac{\pi}{2} \text{ for } n \in Z\right)$$

$$= \frac{(1+\sin x)}{\cos x(1-\sin x)}$$

$$= \frac{1-\sin^2 x}{\cos x(1-\sin x)^2} \qquad 5$$

$$= \frac{\cos x}{(1-\sin x)^2}. \qquad 5$$

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11. (a) Let $a, b \in \mathbb{R}$. Write down the discriminant of the equation $3x^2 - 2(a+b)x + ab = 0$ in terms of a and b, and **hence**, show that the roots of this equation are real.

Let α and β be these roots. Write down $\alpha + \beta$ and $\alpha\beta$ in terms of a and b.

Now, let $\beta = \alpha + 2$. Show that $a^2 - ab + b^2 = 9$ and **deduce** that $|a| \leq \sqrt{12}$, and find b in terms of a.

(b) Let $c(\neq 0)$ and d be real numbers, and let $f(x) = x^3 + 4x^2 + cx + d$. The remainder when $f(x)$ is divided by $(x+c)$ is $-c^3$. Also, $(x-c)$ is a factor of $f(x)$. Show that $c = -2$ and $d = -12$.

For these values of c and d, find the remainder when $f(x)$ is divided by $(x^2 - 4)$.

(a) $3x^2 - 2(a+b)x + ab = 0$

The discriminant $\Delta = 4(a+b)^2 - 12(ab)$ **10**

$$= 4(a^2 + 2ab + b^2 - 3ab)$$

$$= 4(a^2 - ab + b^2)$$

$$= 4\left[\left(a - \frac{b}{2}\right)^2 + \frac{3b^2}{4}\right] \geq 0 \text{ for all } a, b \in \mathbb{R}. \quad 5$$

Hence the roots are real. 5 5

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$\alpha + \beta = \frac{2}{3}(a+b) \quad \alpha\beta = \frac{ab}{3}$ 5

5

$\beta = \alpha + 2 \Rightarrow (\beta - \alpha)^2 = 4$

$\Rightarrow (\beta + \alpha)^2 - 4\alpha\beta = 4$ 5

$\Rightarrow \frac{4}{9}(a+b)^2 - \frac{4}{3}ab = 4$ 5 5 5

$\Rightarrow a^2 + 2ab + b^2 - 3ab = 9$

$\Rightarrow a^2 - ab + b^2 = 9$ 5

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$b^2 - ab + a^2 = 9$

$$\Rightarrow \left(b - \frac{a}{2}\right)^2 = \frac{a^2}{4} - a^2 + 9$$

$$= -\frac{3a^2}{4} + 9$$

$$= \frac{3}{4}(12 - a^2) \quad 10$$

$$\Rightarrow 12 - a^2 \geq 0 \quad 5$$

$$\Rightarrow \quad |a| \leq \sqrt{12} \quad 5$$

$$b = \frac{a}{2} \pm \frac{\sqrt{3}}{2}\sqrt{12 - a^2}. \quad 10$$

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(b) $f(x) = x^3 + 4x^2 + cx + d$

5 $f(-c) = -c^3 + 4c^2 - c^2 + d = -c^3$

$\Rightarrow 3c^2 + d = 0 \quad ----- (1)$ 5

5 $f(c) = c^3 + 4c^2 + c^2 + d = 0$

$\Rightarrow c^3 + 5c^2 + d = 0 ----- (2)$ 5

(2) – (1) gives $c^3 + 2c^2 = 0$ 5

$\Rightarrow c^2(c + 2) = 0$

Since $c \neq 0$, we get $c = -2$. 5

$\Rightarrow d = -3c^2 = -12.$ 5

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Now $f(x) = x^3 + 4x^2 - 2x - 12.$

When $f(x)$ is divided by $x^2 - 4$, the remainder is of the form $\lambda x + \mu$.

That is $f(x) = (x^2 - 4)q(x) + \lambda x + \mu.$ 5

$\Rightarrow f(x) = (x - 2)(x + 2)q(x) + \lambda x + \mu.$

5 $f(2) = 8 = 2\lambda + \mu$ and $f(-2) = 0 = -2\lambda + \mu$ 5

$\Rightarrow \mu = 4$ and $\lambda = 2.$ 5

$\therefore$ remainder $= 2x + 4.$ 5

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12. (a) A committee of six members has to be selected from among the members of two groups, each having three boys and two girls, such that the number of girls in the committee is at most two. Find the number of different such committees that can be formed if,

(i) even number of members from each group are to be selected for the committee,

(ii) only one girl is to be selected for the committee.

(b) Let $f(r)=\dfrac{1}{(r+1)^2}$ and $U_r=\dfrac{(r+2)}{(r+1)^2(r+3)^2}$ for $r\in\mathbb{Z}^+$.

Show that $f(r)-f(r+2)=4U_r$ for $r\in\mathbb{Z}^+$.

Hence, show that $\sum\limits_{r=1}^{n} U_r=\dfrac{13}{144}-\dfrac{1}{4(n+2)^2}-\dfrac{1}{4(n+3)^2}$ for $n\in\mathbb{Z}^+$.

Deduce that the infinite series $\sum\limits_{r=1}^{\infty} U_r$ is convergent and find its sum.

Let $t_n=\sum\limits_{r=n}^{2n} U_r$ for $n\in\mathbb{Z}^+$.

Show that $\lim\limits_{n\to\infty} t_n=0$.

(a) (i)

Different ways of selecting		Number of committees	
Group 1	Group 2		
2	4		
1G 1B	1G 3B	$2\times 3\times 2\times 1=12$	10
2B	1G 3B	${}^3C_2\times 2\times 1=6$	10
2B	2G 2B	${}^3C_2\times {}^2C_2\times {}^3C_2=9$	10
		27	5

$\therefore$ Number of different such committees $=27\times 2$

$=54$ 10 **45**

(ii) 1G 5B

10 ${}^4C_1\times {}^6C_5=24.$ 5 **15**

Aliter for (i)

Group 1		Group 2		Number of committees	
M(3)	F(2)	M(3)	F(2)		
2		2	2	${}^{3}C_2 \times {}^{3}C_2 \times {}^{2}C_2 = 9$	10
2		3	1	${}^{3}C_2 \times {}^{3}C_3 \times {}^{2}C_1 = 6$	10
1	1	3	1	${}^{3}C_1 \times {}^{2}C_1 \times {}^{3}C_3 \times {}^{2}C_1 = 12$	10
2	2	2		9	
3	1	2		6	5
3	1	1	1	12	

Number of committees: $9 + 6 + 12 + 9 + 6 + 12 = 54$ 10

13.(*a*) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2a \\ -1 & 0 \\ 1 & 3a \end{pmatrix}$, where $a \in \mathbb{R}$.

Find the matrix $\mathbf{P}$ defined by $\mathbf{P} = \mathbf{AB}$ and show that $\mathbf{P}^{-1}$ **does not exist** for any value of a.

Show that if $\mathbf{P}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $a = 2$.

With this value for a, let $\mathbf{Q} = \mathbf{P} + \mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

Write down $\mathbf{Q}^{-1}$ and find the matrix $\mathbf{R}$ such that $\mathbf{AA}^{\mathrm{T}} - \frac{1}{2}\mathbf{R} = \left(\frac{1}{5}\mathbf{Q}\right)^{-1}$.

(*b*) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Define the modulus $|z|$ and the conjugate $\bar{z}$ of z.

Show that (i) $z\bar{z} = |z|^2$,

(ii) $z + \bar{z} = 2\,\mathrm{Re}\, z$ and $z - \bar{z} = 2i\,\mathrm{Im}\, z$.

Let $z \neq 1$ and $w = \frac{1+z}{1-z}$. Show that $\mathrm{Re}\, w = \frac{1-|z|^2}{|1-z|^2}$ and $\mathrm{Im}\, w = \frac{2\,\mathrm{Im}\, z}{|1-z|^2}$.

Show further that if $z = \cos\alpha + i\sin\alpha$ $(0 < \alpha < 2\pi)$, then $w = i\cot\frac{\alpha}{2}$.

(*c*) In an Argand diagram, the points A and B represent the complex numbers $-3i$ and 4 respectively. The points C and D lie in the first quadrant such that $ABCD$ is a rhombus and $B\hat{A}D = \theta$, where $\theta = \sin^{-1}\left(\frac{7}{25}\right)$. Find the complex numbers represented by the points C and D.

(a) $P = AB = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2a \\ -1 & 0 \\ 1 & 3a \end{pmatrix} = \begin{pmatrix} 2 & 2a \\ 1 & a \end{pmatrix}.$ 10 | 10

$\begin{vmatrix} 2 & 2a \\ 1 & a \end{vmatrix} = 2a - 2a = 0.$ 5

$\therefore P^{-1}$ does not exist for any value of a. 5 | 10

Aliter

For the existence of P^{-1},

There must exist $b, c, d, e \in \mathbb{R}$ such that

$$\begin{pmatrix} 2 & 2a \\ 1 & a \end{pmatrix}\begin{pmatrix} b & c \\ d & e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 5

$\Leftrightarrow 2b + 2ad = 1,\quad b + ad = 0,\quad 2c + 2ae = 0$ and $c + ae = 1,$

This is a contradiction.

$\therefore P^{-1}$ does not exist for any value of a.

5 | 10

If $P\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} 2+4a \\ 1+2a \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$ 5

$\Leftrightarrow 2 + 4a = 10$ and $1 + 2a = 5.$

$\Leftrightarrow a = 2.$ 5 | 10

$a = 2.$

$Q = P + I = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}.$ 5

$\therefore Q^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}.$ 10 | 15

$AA^T - \frac{1}{2}R = \left(\frac{1}{5}Q\right)^{-1}$

$= 5Q^{-1}.$ 5

$\Leftrightarrow R = 2AA^T - 10Q^{-1}$

$$= 2\begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 0 & -1 \end{pmatrix} - 10\left(\tfrac{1}{5}\right)\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}$$ 5

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$$= 2\begin{pmatrix} 2 & 6 \\ 6 & 21 \end{pmatrix} - \begin{pmatrix} 6 & -8 \\ -2 & 6 \end{pmatrix}.$$

$$= \begin{pmatrix} -2 & 20 \\ 14 & 36 \end{pmatrix}.$$ 5

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(b) $z = x + iy \quad x, y \in \mathbb{R}$

$|z| = \sqrt{x^2 + y^2}$ and $\bar{z} = x - iy$. 5

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(i) $z\,\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

(ii) $z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}\, z$ and 5

$z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2i\,\mathrm{Im}\, z$. 5

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If $z \neq 1$, $w = \dfrac{1+z}{1-z} \times \dfrac{1-\bar{z}}{1-\bar{z}} = \dfrac{1 - z\,\bar{z} + z - \bar{z}}{|1-z|^2} = \dfrac{1 - |z|^2 + 2i\,\mathrm{Im}\, z}{|1-z|^2}$

5 5 5

$\Rightarrow \mathrm{Re}\, w = \dfrac{1-|z|^2}{|1-z|^2}$ and $\mathrm{Im}\, w = \dfrac{2\mathrm{Im}\, z}{|1-z|^2}$ 5

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$z = \cos\alpha + i\sin\alpha \; (0 < \alpha < 2\pi)$.

Then $|z| = 1. \Leftrightarrow \mathrm{Re}\, w = 0$.

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$\therefore\; w = \dfrac{2i\,\mathrm{Im}\, z}{|1-z|^2} = \dfrac{2i\sin\alpha}{(1-\cos\alpha)^2 + \sin^2\alpha} = \dfrac{2i\sin\alpha}{2(1-\cos\alpha)} = i\,\dfrac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}} = i\cot\dfrac{\alpha}{2}$.

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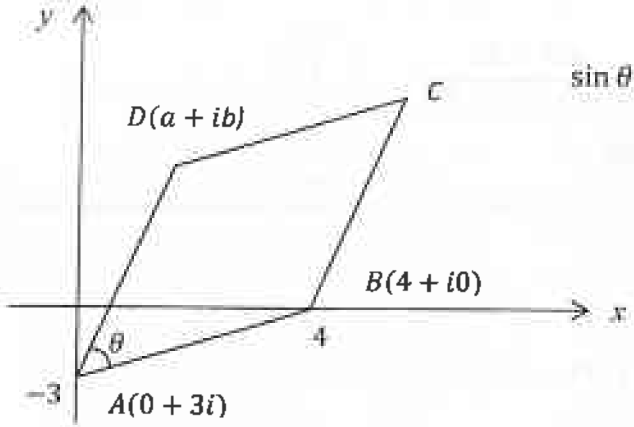
(c)



$\sin\theta = \frac{7}{25}, \left(0 < \theta < \frac{\pi}{2}\right)$

$\Rightarrow \cos\theta = \frac{24}{25}$

Let $D \equiv (a, b)$.

Note that AD can be obtained by rotating AB about A counterclockwise by an angle .

$\therefore a + i(b+3) = (4+3i)(\cos\theta + i\sin\theta)$ 10

$= (4+3i)\left(\frac{24}{25} + i\frac{7}{25}\right)$

$\Leftrightarrow a + i(b+3) = 3 + 4i.$

$\Leftrightarrow a = 3$ and $b = 1$.

$\therefore D$ represents $3 + i$. 05

If $C \equiv (p, q)$, then $\frac{p+0}{2} = \frac{3+4}{2}$ and $\frac{q-3}{2} = \frac{1+0}{2}$.

$\Rightarrow p = 7$ and $q = 4$.

$\therefore C$ represents $7+4i$. 05

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(b)

$$f(r) - f(r+2) = \frac{1}{(r+1)^2} - \frac{1}{(r+3)^2} \quad 5$$

$$= \frac{4(r+2)}{(r+1)^2(r+3)^2} \quad 5$$

$$= 4U_r \quad 5$$

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Then

$r = 1; \quad 4U_1 = f(1) - f(3)$

$r = 2; \quad 4U_2 = f(2) - f(4)$ 10

$r = 3; \quad 4U_3 = f(3) - f(5)$

$\vdots$

$r = n-2; \quad 4U_{n-2} = f(n-2) - f(n)$

$r = n-1; \quad 4U_{n-1} = f(n-1) - f(n+1)$

$r = n; \quad 4U_n = f(n) - f(n+2)$ 10

$$4\sum_{r=1}^{n} U_r = f(1) + f(2) - f(n+1) - f(n+2) \quad 10$$

$$= \frac{1}{4} + \frac{1}{9} - \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2}$$

$$\therefore \sum_{r=1}^{n} U_r = \frac{13}{144} - \frac{1}{4(n+2)^2} - \frac{1}{4(n+3)^2}. \quad 10$$

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The limit of the R. H. S. as $n \to \infty$ *is* $\frac{13}{144}$. 5

$\therefore \sum_{r=1}^{\infty} U_r$ is convergent and the sum is $\frac{13}{144}$. 5

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$$t_n = \sum_{r=n}^{2n} U_r$$

$$= \sum_{r=1}^{2n} U_r - \sum_{r=1}^{n-1} U_r$$ 5

Since $\sum_{r=1}^{\infty} U_r$ is convergent,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{r=1}^{2n} U_r - \lim_{n\to\infty} \sum_{r=1}^{n-1} U_r$$ 5

$$= \frac{13}{144} - \frac{13}{144}$$ 5

$$= 0.$$ 5

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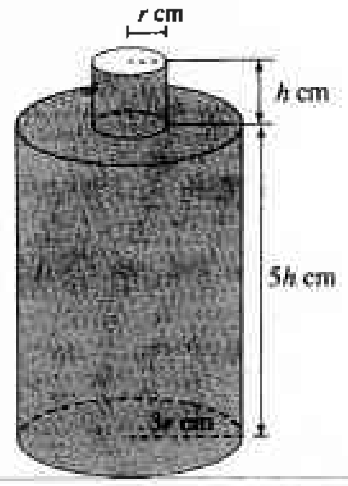
14. (*a*) Let $f(x) = \dfrac{16(x-1)}{(x+1)^2(3x-1)}$ for $x \neq -1, \dfrac{1}{3}$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{-32x(3x-5)}{(x+1)^3(3x-1)^2}$ for $x \neq -1, \dfrac{1}{3}$.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Using the graph, find the values of $k \in \mathbb{R}$ such that the equation $k(x+1)^2(3x-1) = 16(x-1)$ has exactly one root.

(*b*) A bottle with a volume of $391\pi\,\text{cm}^3$ is to be made by removing a disc of radius r cm from the top face of a closed hollow right circular cylinder of radius $3r$ cm and height $5h$ cm, and fixing an open hollow right circular cylinder of radius r cm and height h cm, as shown in the figure. It is given that the total surface area $S\,\text{cm}^2$ of the bottle is $S = \pi r(32h + 17r)$. Find the value of r such that S is minimum.



(a) For $x \neq -1, \frac{1}{3}$; $f(x) = \frac{16(x-1)}{(x+1)^2(3x-1)}$.

Then

$$f'(x) = \frac{16(x+1)^2(3x-1) - 16(x-1)[2(x+1)(3x-1) + 3(x+1)^2]}{(x+1)^4(3x-1)^2}$$ 15

$$= \frac{16(x+1)[(x+1)(3x-1) - 2(x-1)(3x-1) - 3(x-1)(x+1)]}{(x+1)^4(3x-1)^2}$$

$$= \frac{-32x(3x-5)}{(x+1)^3(3x-1)^2} \text{ for } x \neq -1, \frac{1}{3}.$$ 10

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Horizontal asymptotes: $\lim_{x\to\pm\infty} f(x) = 0$, then $y = 0$. 5

$\lim_{x\to -1^{\pm}} f(x) \to \infty$, $\lim_{x \to \frac{1}{3}^-} f(x) \to \infty$ and $\lim_{x \to \frac{1}{3}^+} f(x) \to -\infty$.

Vertical asymptotes: $x = -1$ and $x = \frac{1}{3}$. 5

At turning points $f'(x) = 0$. $\Leftrightarrow$ $x = 0$ or $x = \frac{5}{3}$.

	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \frac{1}{3}$	$\frac{1}{3} < x < \frac{5}{3}$	$\frac{5}{3} < x < \infty$
sign of $f'(x)$	$(+)$	$(-)$	$(+)$	$(+)$	$(-)$
	f is increasing	f is decreasing	f is increasing	f is increasing	f is decreasing
	5	5	5	5	5

There are two turning points: $(0,16)$ is a local minimum and $\left(\frac{5}{3},\frac{3}{8}\right)$ is a local maximum.

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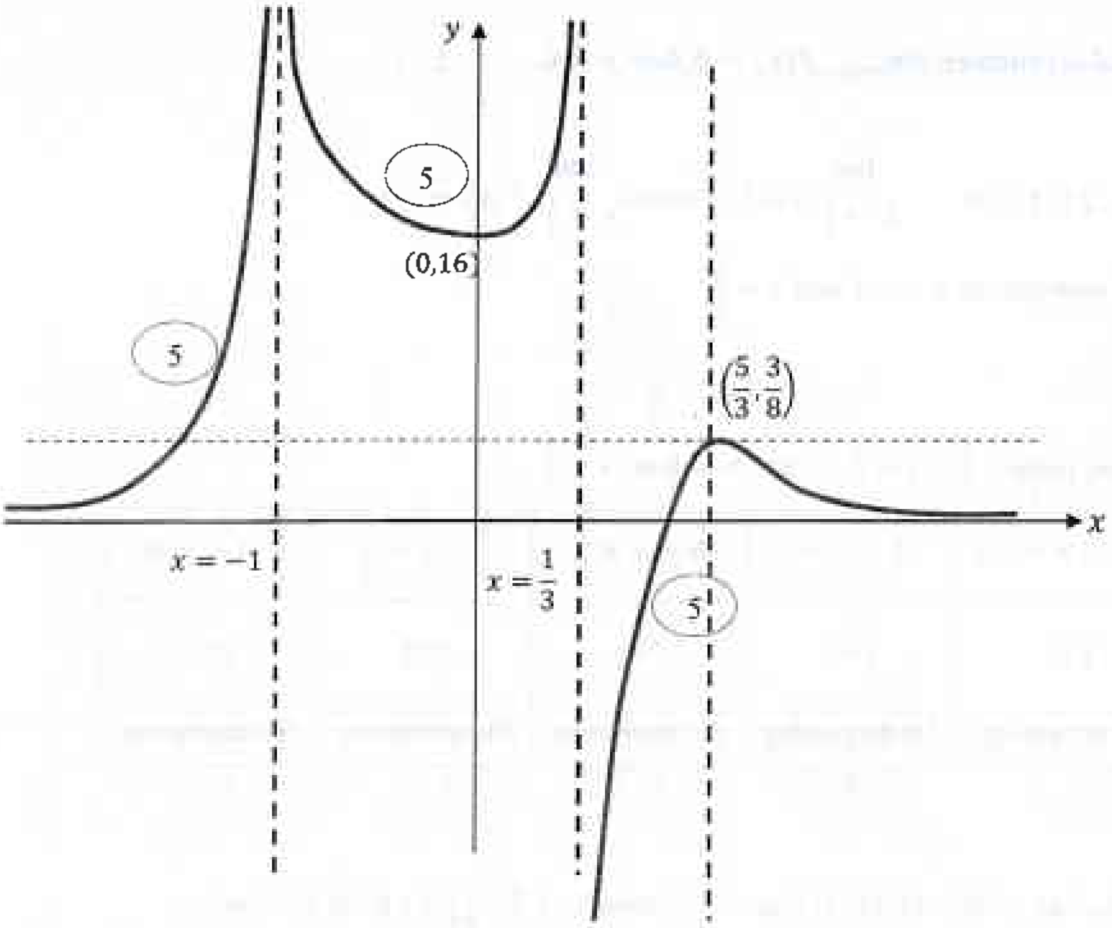

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$k(x+1)^2(3x-1) = 16(x-1).$

$$\Leftrightarrow k = \frac{16(x-1).}{(x+1)^2(3x-1)}. \quad 5$$

The given equation has exactly one root if and only if $k \leq 0$ or $\frac{3}{8} < k < 16.$ 5

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(b) The volume: $391\pi = \pi(3r)^2(5h) + \pi r^2 h$ 10

$\Rightarrow 391 = 46r^2h$

$\Rightarrow h = \frac{17}{2r^2}, \quad (r > 0).$ 5

Surface area: $S = \pi r(32h + 17r)$.

$\Rightarrow \; = 17\pi\left(\frac{16}{r} + r^2\right)$ 5

$$\frac{dS}{dr} = 17\pi\left(-\frac{16}{r^2} + 2r\right) = \frac{34\pi(r^3 - 8)}{r^2}$$ 5 5

$$\frac{dS}{dr} = 0 \Leftrightarrow r = 2.$$ 5

For $0 < r < 2$, $\frac{dS}{dr} < 0$ 5 and $r > 2$, $\frac{dS}{dr} > 0$. 5

$\therefore$ S is minimum when $r = 2$. 5

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15.(a) (i) Comparing the coefficients of x^2, x^1 and x^0, find the values of the constants A, B and C such that $Ax^2(x-1)+Bx(x-1)+C(x-1)-Ax^3=1$ for all $x\in\mathbb{R}$.

Hence, write down $\frac{1}{x^3(x-1)}$ in partial fractions and find $\int\frac{1}{x^3(x-1)}\,dx$.

(ii) Using integration by parts, find $\int x^2\cos 2x\,dx$.

(b) Using the substitution $\theta=\tan^{-1}(\cos x)$, show that $\int_0^{\pi}\frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx=2\ln\left(1+\sqrt{2}\right)$.

Using the formula $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, where a is a constant, find $\int_0^{\pi}\frac{x\sin x}{\sqrt{1+\cos^2 x}}\,dx$.

(a) (i) $Ax^2(x-1)+Bx(x-1)+C(x-1)-Ax^3=1$

Comparing coefficients of powers of x,

$x^2\ :\ -A+B=0$ 5

$x^1\ :\ -B+C=0$ 5

$x^0\ :\ -C=1$ 5

$A=-1, B=-1$ and $C=-1$ 5

$1=-x^2(x-1)-x(x-1)-(x-1)+x^3$ 20

Therefore the required partial fraction is given by

$$\frac{1}{x^3(x-1)}=-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x-1}.$$ 5

Hence, $\int\frac{1}{x^{3(x-1)}}\,dx=-\int\frac{1}{x}\,dx-\int\frac{1}{x^2}\,dx-\int\frac{1}{x^3}\,dx+\int\frac{1}{x-1}\,dx$

$$=-\ln|x|+\frac{1}{x}+\frac{1}{2x^2}+\ln|x-1|+C,$$

where C is an arbitrary constant. 25 30

(ii) $\int x^2 \cos 2x \, dx = \frac{x^2 \sin 2x}{2} - \frac{1}{2}\int 2x \sin 2x \, dx$ 5 5

$= \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{2}\int \cos 2x \, dx$ 5

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$= \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} + C$, where C is an arbitrary constant 5

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(b) $\theta = \tan^{-1}(\cos x);\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan\theta = \cos x \implies \sec^2\theta \, d\theta = -\sin x \, dx$ 5

$x = 0 \implies \theta = \tan^{-1}(1) \implies \theta = \frac{\pi}{4}$ 5

$x = \pi \implies \theta = \tan^{-1}(-1) \implies \theta = -\frac{\pi}{4}$ 5

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$\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos^2 x}} dx = -\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \, d\theta \quad \left(\sqrt{\sec^2\theta} = \sec\theta \text{ as } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec\theta(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)} d\theta$$ 5

$$= \ln|\sec\theta + \tan\theta| \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$ 5

$$= \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1)$$ 5

$$= \ln\left(\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right)$$

$$= 2\ln(\sqrt{2}+1).$$ 5 $\boxed{50}$

$I = \int_0^{\pi} \frac{x \sin x}{\sqrt{1+\cos^2 x}} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{\sqrt{1+\cos^2(\pi - x)}} dx$ 5

$= \pi \int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos^2 x}} dx - \int_0^{\pi} \frac{x \sin x}{\sqrt{1+\cos^2 x}} dx$ 5

$\Rightarrow I = \pi[2\ln(\sqrt{2}+1)] - I$ 5

$\Rightarrow 2I = 2\pi \ln(\sqrt{2}+1)$

$\Rightarrow I = \pi \ln(\sqrt{2+1})$. 5

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16. Let $A \equiv (-2, -3)$ and $B \equiv (4, 5)$. Find the equations of the lines l_1 and l_2 through the point A such that the acute angle made by each of the lines l_1 and l_2 with the line AB is $\frac{\pi}{4}$.

The points P and Q are taken on l_1 and l_2 respectively such that $APBQ$ is a square.

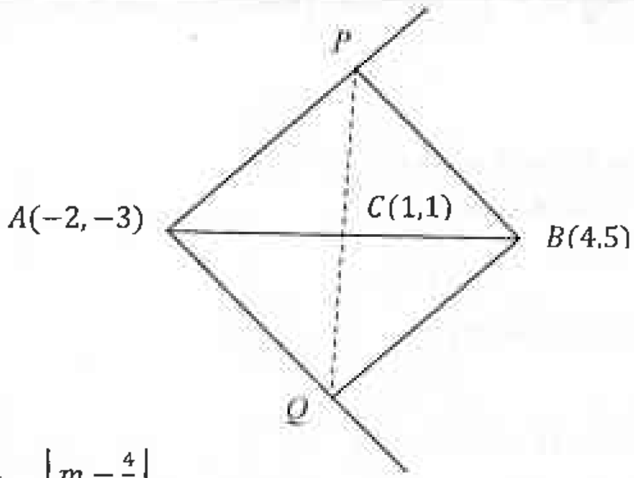
Find the equation of PQ, and the coordinates of P and Q.

Also, find the equation of the circle S through the points A, P, B and Q.

Let $\lambda > 1$. Show that the point $R \equiv (4\lambda, 5\lambda)$ lies outside the circle S.

Find the equation of the chord of contact of the tangents drawn from the point R to the circle S.

As $\lambda\ (> 1)$ varies, show that these chords of contact pass through a fixed point.



$\tan\frac{\pi}{4} = \left|\frac{m - \frac{4}{3}}{1 + \frac{4m}{3}}\right|$ 10

$\Rightarrow \left(m - \frac{4}{3}\right)^2 = \left(1 + \frac{4m}{3}\right)^2$ 5

$\Rightarrow 7m^2 + 48m - 7 = 0$

$\Rightarrow (7m - 1)(m + 7) = 0$

$\Rightarrow m = \frac{1}{7}$ or $m = -7$.

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$\therefore$ The required equations are

(i) $y+3=\frac{1}{7}(x+2) \Longrightarrow x-7y-19=0,$ 10

and

(ii) $y+3=-7(x+2) \Rightarrow 7x+y+17=0.$ 10 45

Let the line $x-7y-19=0$ be l_1 and the other be l_2.

Equation of PQ : $y-1=-\frac{3}{4}(x-1) \Longrightarrow 3x+4y-7=0$ 10

Intersection of line PQ and of l_1 is $P=(5,-2)$ 5

If $Q=(x_0,y_0)$, then

$$\frac{5+x_0}{2}=1 \Rightarrow x_0=-3 \quad 5$$

$$\frac{-2+y_0}{2}=1 \Rightarrow y_0=4$$

$$Q \equiv (-3,4). \quad 5$$

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The circle passing through the points A, P, B and Q is the circle with AB as a diameter. 10

$(y-5)(y+3)+(x-4)(x+2)=0 \Rightarrow x^2+y^2-2x-2y-23=0.$

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Note that $CR^2=(4\lambda-1)^2+(5\lambda-1)^2$ and the radius of the circle is 5. 10

Now $CR^2-25=(4\lambda-1)^2+(5\lambda-1)^2-25$ 5
$=41\lambda^2-18\lambda-23$
$=(\lambda-1)(41\lambda+23)>0$ as $\lambda>1$. 10

Therefore, R lies outside the circle. 5 30

The required chord of contact is given by

$$x(4\lambda)+y(5\lambda)-(x+4\lambda)-(y+5\lambda)-23=0$$
$$(-x-y-23)+\lambda(4x+5y-9)=0 \quad 5 \quad 10$$

Hence the chords of contact pass through the point of intersection of lines

$4x + 5y - 9 = 0$ and $x + y + 23 = 0$. 10

It is a fixed point. 5

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17. (a) Solve $\cos 2\theta + \cos 3\theta = 0$ for $0 \le \theta \le \pi$.

Write down $\cos 2\theta$ and $\cos 3\theta$ in terms of $\cos\theta$, and show that

$$\cos 2\theta + \cos 3\theta = 4t^3 + 2t^2 - 3t - 1, \text{ where } t = \cos\theta.$$

Hence, write down the three roots of the equation $4t^3 + 2t^2 - 3t - 1 = 0$ and show that the roots of the equation $4t^2 - 2t - 1 = 0$ are $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Deduce that $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$.

(b) Let ABC be a triangle and let D be the point on BC such that $BD:DC = m:n$, where $m, n > 0$. It is given that $B\hat{A}D = \alpha$ and $D\hat{A}C = \beta$. Using the Sine Rule for the triangles BAD and DAC, show that $\frac{mb}{nc} = \frac{\sin\alpha}{\sin\beta}$, where $b = AC$ and $c = AB$.

Hence, show that $\frac{mb - nc}{mb + nc} = \tan\left(\frac{\alpha-\beta}{2}\right)\cot\left(\frac{\alpha+\beta}{2}\right)$.

(c) Show that $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{2}$.

(a) For $0 \le \theta \le \pi$, $\cos 3\theta = -\cos 2\theta = \cos(\pi - 2\theta)$

$$3\theta = 2n\pi \pm (\pi - 2\theta),\ n \in \mathbb{Z}.$$ 5

$$5\theta = 2n\pi + \pi,\ n \in \mathbb{Z} \text{ or } \theta = 2n\pi - \pi,\ n \in \mathbb{Z}.$$

Since $0 \le \theta \le \pi$, $\theta = \pi$, $\frac{\pi}{5}$ and $\frac{3\pi}{5}$ are solutions.

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$\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Therefore $\cos 2\theta + \cos 3\theta = 4\cos^3\theta + 2\cos^2\theta - 3\cos\theta - 1$

$= 4t^3 + 2t^2 - 3t - 1$, where $t = \cos\theta$. 10

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$\therefore$ The roots of $4t^3 + 2t^2 - 3t - 1 = 0$ are $\cos\pi, \cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. 10

$\cos\pi = -1 \implies t + 1$ is a factor of $4t^3 + 2t^2 - 3t - 1$.

$\implies 4t^3 + 2t^2 - 3t - 1 = (t+1)(4t^2 - 2t - 1) = 0$ 10

$\implies \cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are roots of $4t^2 - 2t - 1 = 0$. 5

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$$t = \frac{2 \pm \sqrt{2^2 + 4\times4\times1}}{2\times4} = \frac{1\pm\sqrt{5}}{4}$$

$$\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4} \quad \text{as} \cos\frac{3\pi}{5} < 0.$$

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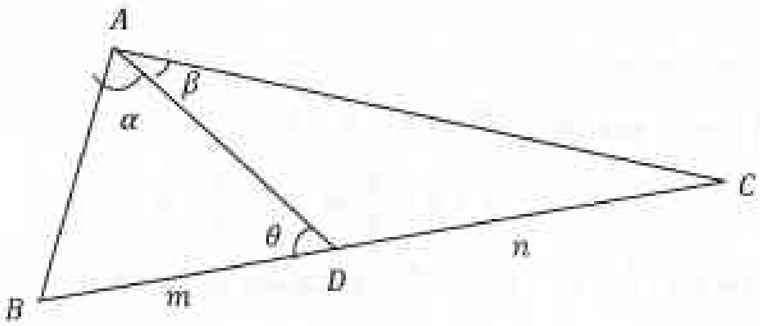
(b)



Let $B\hat{D}A = \theta$.

Using the Sine Rule:

Triangle BAD : $\frac{BD}{\sin\alpha} = \frac{c}{\sin\theta}$ 10

Triangle ADC: $\frac{DC}{\sin\beta} = \frac{b}{\sin(\pi-\theta)}$ 10

$$\implies \frac{(BD)\sin\beta}{(DC)\sin\alpha} = \frac{c}{b}$$

$\implies \frac{mb}{nc} = \frac{\sin\alpha}{\sin\beta}$. 5

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$$mb = nc\frac{\sin\alpha}{\sin\beta}$$

$$\Rightarrow \frac{mb - nc}{mb + nc} = \frac{nc\frac{\sin\alpha}{\sin\beta} - nc}{nc\frac{\sin\alpha}{\sin\beta} + nc}$$ 5

$$= \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$$

$$= \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$ 5 5

$$= \tan\left(\frac{\alpha-\beta}{2}\right)\cot\left(\frac{\alpha+\beta}{2}\right).$$

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(c) Let $\tan^{-1}\left(\frac{1}{3}\right) = \gamma$ and $\tan^{-1}\left(\frac{4}{3}\right) = \delta, \quad 0 < \delta, \gamma < \frac{\pi}{2}.$

5 $$2\gamma + \delta = \frac{\pi}{2} \Leftrightarrow 2\gamma = \frac{\pi}{2} - \delta$$

$\Leftrightarrow \tan(2\gamma) = \tan\left(\frac{\pi}{2} - \delta\right)$ (Since $\frac{\pi}{2} - \delta$ is acute, 2γ is acute)

$$\tan 2\gamma = \frac{2\tan\gamma}{1 - \tan^2\gamma} = \frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$$ 5

$$\tan\left(\frac{\pi}{2} - \delta\right) = \cot\delta = \frac{3}{4}$$ 5

$$\therefore 2\gamma + \delta = \frac{\pi}{2}.$$ 5 $\boxed{20}$
