Provincial Department of Education - Northern Province

Combined Mathematics - I

Grade:13 (2023)	**10 T I**	**Three hours** **Additional Reading Time: 10 minutes**

Index No. | | | | | | |

Instructions

- This question paper consists of two parts;
  Part A (questions 1-10) and **part B**(questions 11- 17).

Part - A

- Answer **all** questions. Answers should be written in the space provided on the questions paper. If additional space needed, you may use additional answer sheets.

Part - B

- Answer only 5 questions.
- After the allocated time hand over the paper to the supervisor with both parts attached together.
- **Only part B** of the paper is allowed to be taken out of the Examination Hall..

(10) Combined Mathematics - I		
Part	Question No.	Marks
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
Total		

Combined Mathematics-I	
Combined Mathematics-II	
Final Marks	

Part A

1. Using the **Principle of Mathematical Induction,** prove that $\sum_{r=1}^{n}(2r+1) = n(n+2)$ for all $n \in \mathbb{Z}^+$.

2. Sketch the graphs of $y = |4 - x|$ and $y = 3\,\big||x| - 2\big|$ in the same diagram. **Hence or otherwise,** find the set of all positive real values of x satisfying the inequality $\big||x| - 2\big| - \frac{1}{3}|x - 4| \leq 0$.

3. Sketch, in the same Argand diagram, the loci of the points that represent the complex number $\bar{z}$ satisfying $|z-4i|=2$ and $Arg\left(\frac{i}{z}\right)=\frac{\pi}{6}$. **Hence,** if the complex number z_0 satisfies both equations, find $|z_0|$.

4. Let $n \in \mathbb{Z}^+$ and $n \geq 4$. Write down the coefficient of x^4 of the expansion of $(1+kx+x^2)(1+x)^n$. **Hence,** show that ${}^{2023}C_4 + 2\,{}^{2023}C_3 + {}^{2023}C_2 = {}^{2025}C_4$.

5. Show that $\lim_{x\to 0} \frac{\tan^2 x}{1-\cos(\tan x)} = 2.$

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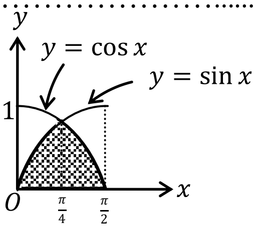
6. The region S enclosed by the curves $y = \sin x$, $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$ and the x-axis is shaded in the figure. The region S is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\frac{\pi}{4}(\pi - 2)$.



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7. A curve C is given parametrically by $x = at^2, y = at^3$ for $0 < t < 1$. Show that the equation of the tangent drawn to the curve C at the point $P(at^2, at^3)$ is $3tx - 2y - at^3 = 0$.
If the tangent line make equal intercepts on x and y axes, find the coordinates of P

8. A Straight line l passing through the point of intersection of the lines $5x + y - 1 = 0$, $3x - 4y + 1 = 0$ and make equal intercepts on positive x and y axes. Find the equation of the line l.

9. Find the equation of the circle passing through the point $(1, 1)$ which has its centre on the line $x + 2y - 4 = 0$ and intersect the circle $x^2 + y^2 - 4 = 0$ orthogonally.

10. Show that $\sec\theta - \tan\theta = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$. **Hence,** find the value of $\tan\frac{\pi}{8}$.

Part B

Answer five questions only.

11. (a) Let $a, b, c \in \boldsymbol{R}$. Show that the equation $3x^2 - 2(a+b+c)x + ab + bc + ca = 0$ has real roots.

If one root is three times the other, show that $(a+b+C)^2 = 4(ab+bc+ca)$.

(b) Let $k \in \boldsymbol{R}$ and $f(x) = 16x^2 + (k+1)x + 1 = 0$.

If the equation $f(x) = 0$ has real root, find the values of k.

Let α and β be the roots of $f(x) = 0$. find the equation in terms of k whose roots are 4α and 4β.

(c) If $(x-1)$ is a factor of the polynomial $2x^3 - 3ax^2 + ax + b$ and the remainder when the polynomial is divided by $(x+2)$ is -54, find the values of a and b.

For these values of a and b, express the polynomial as the product of linear factors.

Hence, factorise the following polynomials

(i) $2x^6 - 9x^4 + 3x^2 + 4$, (ii) $4x^3 + 3x^2 - 9x + 2$.

12. (a) Five identical red balls, three identical blue balls and two identical white balls are to distributed among five students such that each student has at least one ball and no balls are left over. Find the number of different ways in which

(i) three students get one blue ball each and out of the remaining two students one gets five red balls and the other gets two white balls.

(ii) a particular student gets five red balls.

(iii) all students get two balls each but no one gets both a blue ball and a white ball.

(b) Let $U_r = \frac{(-1)^r}{(2r-1)(2r+3)}$ and $f(r) = \frac{A}{(2r-1)} + \frac{B}{(2r+1)}$ for $r \in \mathbb{Z}^+$, where A and B are real constants.

Find the values of A and B such that $\frac{1}{(2r-1)(2r+3)} = f(r) + f(r+1)$ for $r \in \mathbb{Z}^+$.

Hence or otherwise, show that $\sum_{r=1}^{n} U_r = -\frac{1}{6} + \frac{(-1)^n}{4}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$ for $r \in \mathbb{Z}^+$.

Show that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find the sum.

Find $\sum_{r=1}^{\infty}(U_r + 5U_{r+1})$.

13. (a) Let $x \neq y,\ y \neq 0$ and $\boldsymbol{B} \equiv \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$ be a 2×2 matrix.

(i) Show that $\boldsymbol{B}^{-1}$ exists.

(ii) Write down $\boldsymbol{B}^{-1}$.

(iii) If $\boldsymbol{B}^3 = \boldsymbol{I}$, show that $x = -2$ and $y = -3$, where $\boldsymbol{I}$ is the identity matrix of order 2.

(iv) For $x = -2$ and $y = -3$, verify that $\boldsymbol{B}^{-1} + \boldsymbol{B} + \boldsymbol{I} = \boldsymbol{O}$, where $\boldsymbol{O}$ is a 2×2 zero matrix.

(v) For $x = -2$ and $y = -3$, find $\boldsymbol{I} + \boldsymbol{B} + \boldsymbol{B}^2 + \boldsymbol{B}^3 + \boldsymbol{B}^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \boldsymbol{B}^{100}$.

(b) Let $z \in \mathbb{C}$. Define $\bar{z}$ and $|z|$ and show that $z\bar{z} = |z|^2$.

Let $a, b, c \in R$ such that $c \neq 1$, $a^2 + b^2 + c^2 = 1$ and $z = \frac{a + i\,b}{1-c}$.

(i) Show that $|z|^2 = \frac{1+c}{1-c}$. **Hence,** express c in terms of z and $\bar{z}$.

(ii) Find $z + \bar{z}$ and $z - \bar{z}$ in terms of a , b and c. **Hence,** express a and b in terms of of z and $\bar{z}$.

(iii) If $|z| = \sqrt{3}$ and $Arg(z) = \tan^{-1}(\sqrt{2})$, find a, b and c.

(c) Let $\theta \neq 2n\pi \pm \frac{\pi}{2}$ for $n \in \mathbb{Z}$ and $z = \cos\theta + i\sin\theta$.

Using the **De Moivre's theorem,** show that $(1 + i\tan\theta)^4 + (1 - i\tan\theta)^4 = \frac{2\cos 4\theta}{\cos^4\theta}$.

Hence, show that $z = i\tan\frac{\pi}{8}$ is a root of $(1+z)^4 + (1-z)^4 = 0$ and find the other roots in the form $i\tan\alpha$, where $0 < \alpha < \pi$.

14. (a) Let $f(x) = 1 + \frac{x+1}{x^2}$ for $x \neq 0$.

Show that $f'(x) = -\frac{x+2}{x^3}$ for $x \neq 0$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Also, find the coordinates of the turning point.

It is given that $f''(x) = \frac{2(x+3)}{x^4}$ for $x \neq 0$.

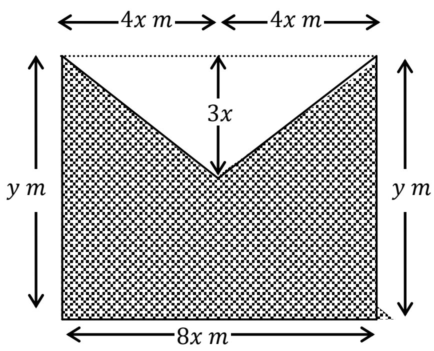
Find the coordinates of the point of inflection of the graph of $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, the turning point and the point of inflection.

State the smallest value of k for which $f(x)$ is one-one on $[k, 0)$.

(b) The shaded region shown in the figure is of area $84\ m^2$. It is obtained by removing a triangle of base $8x\ m$ and height $3x\ m$ from a rectangle of length $y\ m$ and breadth $8x\ m$. Show that the perimeter $L\ m$ of the shaded region is given by $L = 21x + \frac{21}{x}$ for $0 < x \leq \sqrt{7}$.

Find the value of x such that L is minimum.



15. (a) Express $\frac{x^2+x+1}{(x+1)(x^2+1)}$ in partial fractions.

Hence, find the values of the constants p and q such that $\int_0^1 \frac{x^2+x+1}{(x+1)(x^2+1)}\,dx = \frac{q}{p^2}\ln p + \frac{\pi}{p^3}$.

(b) Find $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$.

Using integration by parts, show that $\int_0^{\frac{\pi}{2}} \sin^{2k+2} x\,dx = \frac{2k+1}{2k+2}\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx$.

Using the **Principle of Mathematical Induction** show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{(2n)!\,\pi}{2^{2n+1}.(n!)^2}$.

(c) Show that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

Hence, show that $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx = \frac{1}{2}$.

16. A circle s_2 with centre $A(-7, 0)$ touches a circle $s_1 \equiv x^2 + y^2 - 16x - 36 = 0$ externally.

(a) Find the equation of the circle s_2.

(b) Find the equations of the common tangents to the circles s_1 and s_2.

(c) A point $P(h, k)$ is on the second quadrant. A circle s_3 with centre $B(-7, 40)$ has a radius equal to radius of s_2

(i) If a circle with centre P touches the circles s_1 and s_2 externally, show that $8h^2 - k^2 - 8h - 48 = 0$.

(ii) If a circle with centre P touches the circles s_2 and s_3 externally, find the locus of P.

(iii) If a circle s with centre P touches the circles s_1, s_2 and s_3 externally, find the equation of the circle s.

17. (a) (i) If $\tan x = \lambda \tan y$, show that $\sin(x + y) = (\lambda + 1)\cos x \sin y$.

Hence, show that $(\lambda + 1)\sin(x - y) = (\lambda - 1)\sin(x + y)$.

(ii) If $\tan\left(\theta + \frac{\pi}{4}\right) = k\tan\left(\theta + \frac{\pi}{12}\right)$, show that $\sin\left(2\theta + \frac{\pi}{3}\right) = \frac{k+1}{2(k-1)}$.

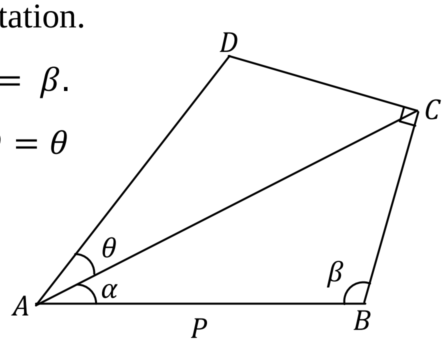
Solve the equation

$(\sqrt{2} - 1)\tan\left(\theta + \frac{\pi}{4}\right) = (\sqrt{2} + 1)\tan\left(\theta + \frac{\pi}{12}\right)$, where $0 \leq \theta \leq \pi$.

(b) State the **Sine Rule** for a triangle ABC, in the usual notation.

In a triangle ABC, $AB = p$, $C\hat{A}B = \alpha$ and $C\hat{B}A = \beta$.

A point D is outside the triangle ABC such that $C\hat{A}D = \theta$ and $B\hat{C}D = \frac{\pi}{2}$ as shown in the figure.



Using the **Sine Rule** for suitable triangles show that

(i) $\frac{AC}{\cos(\alpha+\beta+\theta)} = -\frac{CD}{\sin\theta} = \frac{AD}{\cos(\alpha+\beta)}$ and

(ii) $\frac{AC}{\sin\beta} = \frac{p}{\sin(\alpha+\beta)} = \frac{BC}{\sin\alpha}$.

If $C\hat{B}D = \frac{\pi}{4}$, show that $\cos(\alpha + \beta + \theta) = -\frac{\sin\theta\ \sin\beta}{\sin\alpha}$ and $AD = -\frac{p\sin\alpha}{\sin\theta\tan(\alpha+\beta)}$.

(c) Solve $\tan(\tan^{-1}3x - \tan^{-1}2) + \tan(\tan^{-1}3 - \tan^{-1}2x) = \frac{3}{8}$.