

**G.C.E. (A.L.) Support Seminar - 2016**  
**Combined Mathematics - Paper I**  
**Answer Guide**

**Part A**

1. Let  $f(n) = 4^n + 15n - 1$ ;  $n \in \mathbb{Z}^+$

When  $n = 1$ ,  $f(1) = 4 + 15 - 1 = 18 = 9 \times 2$

$\therefore f(1)$  is divisible by 9.

$\therefore$  The statement is true when  $n = 1$ . (5)

Let us assume that the given expression is divisible by 9 when  $n = p$ ,  $p \in \mathbb{Z}^+$

That is,  $f(p) = 4^p + 15p - 1 = 9k$ ;  $k \in \mathbb{Z}^+$ . (5)

$$f(p+1) = 4^{p+1} + 15(p+1) - 1$$

$$= 4 \cdot 4^p + 15p + 15 - 1$$

$$= 4 [9k - 15p + 1] + 15p + 15 - 1 \quad (5)$$

$$= 4 \times 9k - 45p + 18$$

$$= 9 [4k - 5p + 2]$$

$$= 9 \lambda ; \lambda = 4k - 5p + 2 \in \mathbb{Z}^+$$

$\therefore f(p+1)$  is divisible by 9.

$\therefore$  The statement is true when  $n = p + 1$  (5)

$\therefore$  By the Principle of Mathematical Induction, the given expression is divisible by 9 for all positive integers  $n$ . (5)

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2.

$$\left( \frac{1}{2} + 7^{\frac{1}{5}} \right)^{10} = \sum_{r=0}^{10} {}^{10}C_r \left( \frac{1}{2} \right)^{10-r} \left( 7^{\frac{1}{5}} \right)^r$$

$$T_r = {}^{10}C_{r-1} \left( \frac{1}{2} \right)^{\frac{11-r}{2}} \left( 7^{\frac{1}{5}} \right)^{\frac{r-1}{5}} ; \text{ here } 1 \leq r \leq 11. \quad (5)$$

Since 2 and 7 are primes, for a term to be rational  $11 - r = 2p$  and  $r - 1 = 5q$ ;

where  $p, q \in \mathbb{Z}^+$ . (5)

That is  $r \in \{1, 3, 5, 7, 9, 11\} \cap \{1, 6, 11\}$  (5)

$\therefore r = 1$  or  $11$ . (5)

$$\therefore \text{The sum of the rational terms} = {}^{10}C_0 2^5 + {}^{10}C_{10} 7^2 = 32 + 49 = 81$$

(5)

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3. The number of ways in which a group of 5 can be formed without restriction

$$= {}^{14}C_5$$

$$= 2002 \quad (5)$$

The number of groups with 5 boys

$$= {}^8C_5$$

$$= 56 \quad (5)$$

The number of groups with 5 girls

$$= {}^6C_5$$

$$= 6 \quad (5)$$

∴ The number of ways in which a group of 5 can be formed such that both sexes are represented in the group

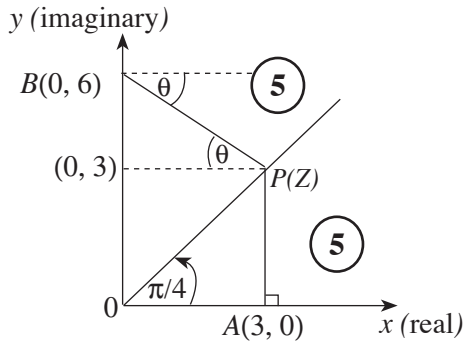
$$= {}^{14}C_5 - ({}^8C_5 + {}^6C_5)$$

$$= 2002 - 56 - 6 \quad (5)$$

$$= 1940 \quad (5)$$

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4.



As indicated in the figure, the point which corresponds to the complex number  $Z = Z_0$  such that  $\text{Arg } Z = \frac{\pi}{4}$  and  $\text{Arg } (Z - 3) = \frac{\pi}{2}$  is P. (5) According to the figure  $\theta = \frac{\pi}{4}$  (5)

$$\therefore \text{Arg } (Z_0 - 6i) = \frac{7\pi}{4} \quad (5)$$

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5.

$$\lim_{x \rightarrow 0} \frac{(1+kx)^2 - (1-kx)^2}{1+k^2x - 1-k^2x}$$

$$= \lim_{x \rightarrow 0} \frac{1+2kx+k^2x^2 - 1+2kx-k^2x^2}{(1+k^2x) - (1-k^2x)} \times \left( \frac{1+k^2x}{1+k^2x} + \frac{1-k^2x}{1-k^2x} \right) \quad (10)$$

$$= \lim_{x \rightarrow 0} \frac{4kx}{2k^2x} \left( \frac{1+k^2x}{1+k^2x} + \frac{1-k^2x}{1-k^2x} \right) ; k, x \neq 0$$

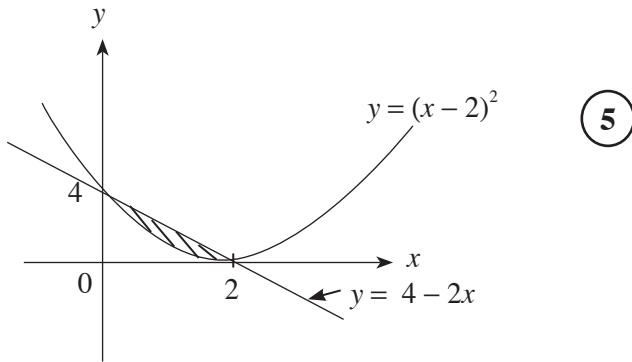
$$= \frac{2}{k} \lim_{x \rightarrow 0} \left( \frac{1+k^2x}{1+k^2x} + \frac{1-k^2x}{1-k^2x} \right) = \left( \frac{2}{k} \right) \times 2 = \frac{4}{k} \quad (5)$$

$$\frac{4}{k} = 1 \quad (5)$$

$$\therefore k = 4 \quad (5)$$

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6.



$$\begin{aligned}
 \text{Area} &= \int_0^2 \{ (4 - 2x) - (x - 2)^2 \} dx && \textcircled{5} \\
 &= \int_0^2 (4 - 2x) dx - \int_0^2 (x - 2)^2 dx \\
 &= \left[ 4x - \frac{2x^2}{2} \right]_0^2 - \left[ \frac{(x-2)^3}{3} \right]_0^2 && \textcircled{5} \\
 &= (8 - 4) - \left[ 0 + \frac{8}{3} \right] && \textcircled{5} \\
 &= 4 - \frac{8}{3} \\
 &= \frac{4}{3} && \textcircled{5}
 \end{aligned}$$

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7. Differentiating with respect to  $t$ ,

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3at^2 - 2t \qquad \textcircled{5}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = (3at^2 - 2t) \cdot \frac{1}{2t} = \frac{3at - 2}{2} ; t \neq 0 \qquad \textcircled{5}$$

$$\left( \frac{dy}{dx} \right)_{t=1} = \frac{3a - 2}{2} , \quad \left( \frac{dy}{dx} \right)_{t=-1} = \frac{-3a - 2}{2} \qquad \textcircled{5}$$

Since the tangents are perpendicular to each other,

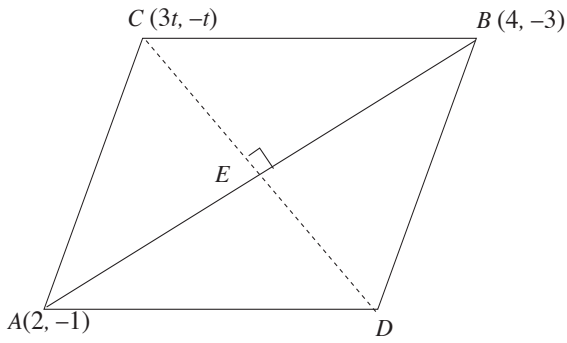
$$\left( \frac{3a - 2}{2} \right) \left( \frac{-3a - 2}{2} \right) = -1 \qquad \textcircled{5}$$

$$9a^2 - 4 = 4 \Rightarrow a^2 = 8/9$$

$$\text{Since } a > 0, a = \frac{2\sqrt{2}}{3} \qquad \textcircled{5}$$

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8.



$$E = (3, -2)$$

Since  $AB$  is perpendicular to  $CE$ ,

$$m_{AB} \times m_{CE} = -1.$$

$$\therefore -1 \times \left(\frac{-2+t}{3-3t}\right) = -1 \quad (5)$$

$$\Rightarrow t = \frac{5}{4} \quad (5)$$

$$\therefore C = \left(\frac{15}{4}, -\frac{5}{4}\right) \quad (5)$$

Let  $D = (\bar{x}, \bar{y})$ . Then,

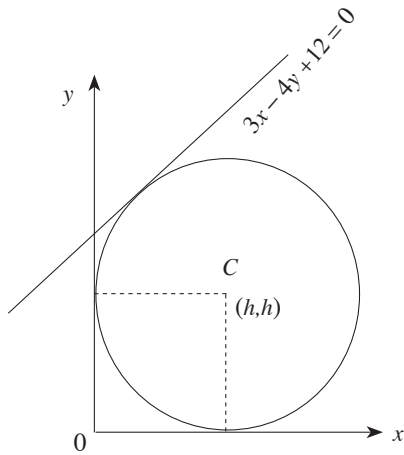
$$\bar{x} = 2 \times 3 - 3t = 6 - 3 \times \frac{5}{4} = \frac{9}{4} \quad (5)$$

$$\bar{y} = 2 \times -2 + t = -4 + \frac{5}{4} = -\frac{11}{4}$$

$$\therefore D = \left(\frac{9}{4}, -\frac{11}{4}\right) \quad (5)$$

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9.



Let  $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$  be a required circle.

Since the circle touches the  $x$  and  $y$  axes,  $C = (h, h)$

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Furthermore, since the line  $3x - 4y + 12 = 0$  touches the circle,

$$\frac{|3h - 4h + 12|}{\sqrt{3^2 + 4^2}} = |h|$$

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$$|-h + 12| = 5|h|$$

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$$\Leftrightarrow (-h + 12) = \pm 5h$$

$$\therefore h = -3 \text{ or } h = 2.$$

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$\therefore$  the equations of the circles are

$$(x - 2)^2 + (y - 2)^2 = 2^2$$

$$(x + 3)^2 + (y + 3)^2 = 3^2$$

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10.  $\cot \alpha - \tan \alpha$

$$= \frac{1}{\tan \alpha} - \tan \alpha$$

$$= \frac{1 - \tan^2 \alpha}{\tan \alpha}$$

$$= \frac{2(1 - \tan^2 \alpha)}{2 \tan \alpha} = \frac{2}{\tan 2\alpha}$$

$$= 2 \cot 2\alpha$$

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$$\cot \alpha - \tan \alpha = 2 \cot 2\alpha \quad \text{--- (1)}$$

$$\therefore \cot 2\alpha - \tan 2\alpha = 2 \cot 4\alpha \quad \text{--- (2)}$$

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$$\cot 4\alpha - \tan 4\alpha = 2 \cot 8\alpha \quad \text{--- (3)}$$

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$$\text{From (1) + 2} \times \text{(2) + 4} \times \text{(3)}$$

$$\cot \alpha - \tan \alpha - 2 \tan 2\alpha - 4 \tan 4\alpha = 8 \cot 8\alpha$$

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$$\cot \alpha = \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha$$

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Part B

11. (a)  $ax^2 + bx + c = 0$

$a \left[ x^2 + \frac{bx}{a} + \frac{c}{a} \right] = 0$

$a \left[ \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \right] = 0$

$a \left[ \left( x + \frac{b}{2a} \right)^2 - \frac{(b^2 - 4ac)}{4a^2} \right] = 0$  (10)

The condition to have coincident roots is  $b^2 - 4ac = 0$ . (10)

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$\frac{a}{x+c} + \frac{b}{x-c} = \frac{k}{2x}$

$\frac{a(x-c) + b(x+c)}{x^2 - c^2} = \frac{k}{2x}$

$x^2[k - 2a - 2b] - 2(bc - ac)x - kc^2 = 0$  (10)

For coincident roots,

$4(bc - ac)^2 - 4(k - 2a - 2b)(-kc^2) = 0$  (10)

That is,  $k^2 - 2(a+b)k + (b-a)^2 = 0$  (5)

If  $k_1$  and  $k_2$  are the roots of the above equation, then

$k_1 + k_2 = 2(a+b)$  (5)  $k_1 k_2 = (b-a)^2$  (5)

$(k_1 - k_2)^2 = (k_1 + k_2)^2 - 4k_1 k_2$  (10)

$= 4(a+b)^2 - 4(b-a)^2$

$= 16ab$

$\therefore |k_1 - k_2| = 4 \sqrt{ab}$  (10)

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(b)  $f(x) = (\lambda + 1)x^2 + (6 - 3\lambda)x + (20 - 12\lambda)$

(i)  $f(x)$  is linear when  $\lambda = -1$ . (5)

(ii) Let the roots be  $\alpha$  and  $-\alpha$ . (5)

Then  $\alpha + (-\alpha) = -\frac{(6 - 3\lambda)}{(\lambda + 1)}$  (5)

$\therefore 0 = 6 - 3\lambda$ . Hence  $\lambda = 2$ . (5)

(iii)  $f(x) = h - b(x - a)^2 = h - b(x^2 - 2ax + a^2) = -bx^2 + 2abx + (h - ba^2)$

$f(x) = (\lambda + 1)x^2 + (6 - 3\lambda)x + (20 - 12\lambda)$  (5)

$\therefore$  by comparing the coefficients,  $-b = \lambda + 1 \Rightarrow b = -(\lambda + 1)$  (1) (5)

$2ab = 6 - 3\lambda \Rightarrow a = -\frac{3(2 - \lambda)}{2(\lambda + 1)}$  (2) (5)

$h - ba^2 = 20 - 12\lambda \Rightarrow h = 4(5 - 3\lambda) - \frac{9(2 - \lambda)^2}{4(\lambda + 1)}$  (3) (10)

Since the maximum value of  $f(x)$  occurs at  $x = 2$ , we obtain that  $a = 2$ . (5)

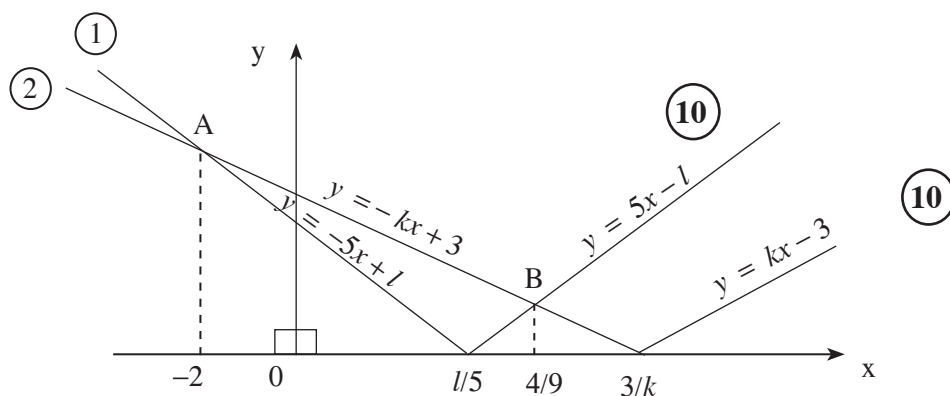
(2)  $\Rightarrow 4(\lambda + 1) = -(6 - 3\lambda) \Rightarrow 4\lambda + 4 = -6 + 3\lambda \Rightarrow \lambda = -10$  (5)

$h = 4(5 + 30) - \frac{9(2 + 10)^2}{4(-10 + 1)}$  (10)

$\therefore$  the maximum value of  $f(x) = 176$  (10)

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12. (a) Since the solution set of the inequality  $|l - 5x| < |kx - 3|$  is  $\{x \mid -2 < x < 4/9\}$ , the two graphs are as illustrated below



(1)  $y = |l - 5x|$

(2)  $y = |kx - 3|$

For point A:  $l + 10 = 2k + 3$  (5)  
 $l - 2k = -7$  (i) (5)

For point B:  $-l + 5 \cdot \frac{4}{9} = -k \cdot \frac{4}{9} + 3$  (5)

$-9l + 4k = 7$  (ii) (5)

by (i) and (ii)  $l = 1$ , (5)  $k = 4$  (5)

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$$(b) \quad S_n = \frac{3n}{2n+1} \quad (5)$$

$$\lim_{n \rightarrow \infty} S_n = \frac{3}{2} \quad (5)$$

The limit is finite.  $(5)$

Therefore, the series is convergent.  $(5)$

$$U_r = S_r - S_{r-1} \quad (5)$$

$$= \frac{3r}{2r+1} - \frac{3(r-1)}{2r-1} \quad (5)$$

$$U_r = \frac{3}{4r^2-1} \quad (5)$$

$$\text{Let } S'_n = \sum_{r=1}^n r^2 \frac{3}{4r^2-1} \quad (5)$$

$$= \sum_{r=1}^n \frac{\frac{3}{4}(4r^2-1) + \frac{3}{4}}{(4r^2-1)} \quad (5)$$

$$= \sum_{r=1}^n \frac{3}{4} + \frac{1}{4} \sum_{r=1}^n \frac{3}{4r^2-1} \quad (10)$$

$$= \frac{3n}{4} + \frac{1}{4} S_n \quad (5)$$

$$\begin{aligned} & (5) \\ & = \frac{3n}{4} + \frac{1}{4} \frac{3n}{(2n+1)} \quad (5) \end{aligned}$$

$$= \frac{3n}{4} \left\{ 1 + \frac{1}{2n+1} \right\} \quad (5)$$

$$= \frac{3n(n+1)}{2(2n+1)} \quad (5)$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \sum_{r=1}^n r^2 U_r = \lim_{n \rightarrow \infty} \frac{3n}{4} \left\{ 1 + \frac{1}{2n+1} \right\} \quad (5)$$

$$= \infty \quad (5)$$

The limit is not finite.  $(5)$

$\therefore$  the series is not convergent.  $(5)$



13. (a)  $\det A = \begin{vmatrix} 3 & p \\ -2 & -3 \end{vmatrix} = -9 + 2p$  (5)

$A^{-1}$  exists only if  $\det A \neq 0$ .

That is, if  $p \neq 9/2$  (5)

$$A^{-1} = \frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} \quad (5)$$

$$A^{-1} = A$$

$$\frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & p \\ -2 & -3 \end{bmatrix} \quad (5)$$

Comparing the corresponding elements,

$$-\frac{3}{2p-9} = 3 \quad \frac{-p}{2p-9} = p \quad (5)$$

$$\frac{2}{2p-9} = -2, \quad \frac{3}{2p-9} = -3 \quad (5)$$

$$(5)$$

$$(5)$$

$$\Rightarrow 2p-9 = -1 \text{ and } p[1+2p-9] = 0$$

Since  $p \neq 0$  we have that  $p = 4$  (5)

Therefore,  $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

$$A^{-1} = A$$

$$\Rightarrow AA^{-1} = A \cdot A = A^2 \quad (5)$$

$$\therefore I = A^2$$

$$\Rightarrow 0 = A^2 - I$$

$$\Rightarrow 0 = (A - I)(A + I); I^2 = I \quad (5)$$

This is of the form,  $0 = BC$ ,

where  $B = A - I = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  (5)

$$= \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$(5)$$

and

$$\begin{aligned}
C = A + I &= \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&\textcircled{5} \\
&= \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \textcircled{5}
\end{aligned}$$

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(b) (i) Let  $Z = x + iy$ , where  $x, y \in \mathbb{R}$   $\textcircled{5}$

$$\begin{aligned}
Z \bar{Z} &= (x + iy)(x - iy) \\
&= x^2 + y^2 \textcircled{5}
\end{aligned}$$

$$= (\overline{x^2 + y^2})^2 = |Z|^2$$

$$\therefore Z \bar{Z} = |Z|^2 .$$

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(ii) Let  $Z_1 = x_1 + iy_1$  and  $Z_2 = x_2 + iy_2$  where  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ .

$$\begin{aligned}
Z_1 Z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
&= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\
&= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \textcircled{5}
\end{aligned}$$

$$\begin{aligned}
\therefore \overline{Z_1 Z_2} &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2) \\
&= x_1(x_2 - iy_2) - iy_1(-iy_2 + x_2) \\
&= (x_1 - iy_1)(x_2 - iy_2) \textcircled{5}
\end{aligned}$$

$$\overline{Z_1 Z_2} = \bar{Z}_1 \bar{Z}_2$$

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(iii)  $\left| \frac{\bar{Z}_1 - 2\bar{Z}_2}{2 - Z_1 \bar{Z}_2} \right| = 1$

$$\Rightarrow |\bar{Z}_1 - 2\bar{Z}_2| = |2 - Z_1 \bar{Z}_2| \textcircled{5}$$

$$\Rightarrow |\bar{Z}_1 - 2\bar{Z}_2|^2 = |2 - Z_1 \bar{Z}_2|^2 \textcircled{5}$$

$$\Rightarrow (\bar{Z}_1 - 2\bar{Z}_2) \overline{(\bar{Z}_1 - 2\bar{Z}_2)} = (2 - Z_1 \bar{Z}_2) \overline{(2 - Z_1 \bar{Z}_2)} \textcircled{5}$$

$$\Rightarrow (\bar{Z}_1 - 2\bar{Z}_2) (Z_1 - 2Z_2) = (2 - Z_1 \bar{Z}_2) (2 - \bar{Z}_1 Z_2) \textcircled{5}$$

$$Z_1 \bar{Z}_1 - 2 \bar{Z}_1 Z_2 - 2 \bar{Z}_2 Z_1 + 4 Z_2 \bar{Z}_2 = 4 - 2 \bar{Z}_1 Z_2 - 2 Z_1 \bar{Z}_2 + Z_1 \bar{Z}_1 Z_2 \bar{Z}_2 \quad (5)$$

$$|Z_1|^2 + 4|Z_2|^2 = 4 + |Z_1|^2 |Z_2|^2$$

$$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2 \cdot |Z_2|^2 - 4 = 0$$

$$|Z_1|^2 (1 - |Z_2|^2) - 4 (1 - |Z_2|^2) = 0$$

$$(1 - |Z_2|^2) (|Z_1|^2 - 4) = 0 \quad (5)$$

Since  $|Z_2| \neq 1$ ,  $|Z_1|^2 - 4 = 0$

$$\therefore |Z_1|^2 = 4$$

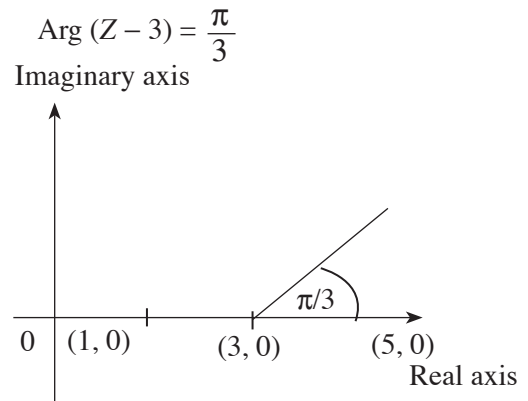
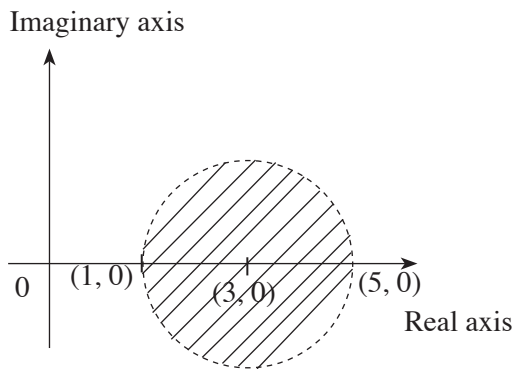
Since  $|Z_1| > 0$ ,  $|Z_1| = 2$

(5)

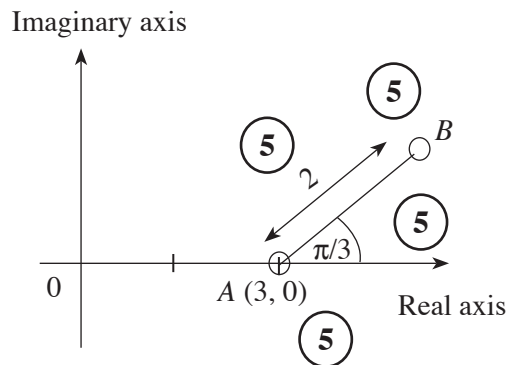
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(c)

$$|Z - 3| < 2$$



$\therefore$  The locus of the point  $P$  which represents the complex number  $Z$  such that  $|Z - 3| < 2$  and  $\text{Arg}(Z - 3) = \frac{\pi}{3}$ ;



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14. (a)  $y = (\sin x)^x \quad 0 \leq x \leq \frac{\pi}{2}$

$\ln y = x \ln |\sin x|$  (10)

$\frac{1}{y} \frac{dy}{dx} = \ln |\sin x| + x \cot x$  (10)

$\therefore \frac{dy}{dx} = [x \cot x + \ln (\sin x)] (\sin x)^x$  (5)

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(b) Volume of the tank =  $\pi x^2 y + \frac{2}{3} \pi x^3$

$\therefore \pi x^2 y + \frac{2}{3} \pi x^3 = 45\pi$  (5)

$\therefore 45 = x^2(y + \frac{2}{3} x)$

$y = \frac{45}{x^2} - \frac{2}{3}x$  (5)

Surface area of the tank

$A = 2\pi x^2 + \pi x^2 + 2\pi xy$  (10)

$A = 3\pi x^2 + 2\pi xy$

$A = 3\pi x^2 + 2\pi x \left( \frac{45}{x^2} - \frac{2}{3}x \right)$

$A = 3\pi x^2 + \frac{90\pi}{x} - \frac{4\pi}{3}x^2$

$A = \frac{5\pi}{3}x^2 + \frac{90\pi}{x}$  (5)

$\frac{dA}{dx} = \frac{10\pi x}{3} - \frac{90\pi}{x^2}$  (5)

$= \frac{10\pi (x^3 - 27)}{3x^2}$

$= \frac{10\pi}{3x^2} (x-3)(x^2+3x+9)$  (5)

$\frac{dA}{dx} = 0$  when  $x = 3$ . (5)

$x$	$0 < x < 3$	$3 < x$
$\frac{dA}{dx}$	$< 0$	$> 0$

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$\therefore$  The surface area is minimum when  $x = 3$ . (5)

$y = \frac{45}{9} - \frac{6}{3}$

$= 3$  (5)

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(c)  $f(x) = \frac{a}{(x-1)^2} + \frac{b}{(x+1)}$

Since  $f(0) = 2$ ,

$a + b = 2$  ——— (1) (5)

$f'(x) = -\frac{2a}{(x-1)^3} - \frac{b}{(x+1)^2}$  (5)

Since  $f'(0) = 0$ ,

$2a - b = 0$  ——— (2) (5)

Form (1) and (2),  $a = \frac{2}{3}$ ,  $b = \frac{4}{3}$  (5)

$f'(x) = -\frac{4}{3} \frac{1}{(x-1)^3} - \frac{4}{3(x+1)^2}$  (5)

$= -\frac{4}{3} \left\{ \frac{(x+1)^2 + (x-1)^3}{(x-1)^3(x+1)^2} \right\}$

$= -\frac{4}{3} \left[ \frac{x^3 - 2x^2 + 5x}{(x-1)^3(x+1)^2} \right]$

$= -\frac{4x}{3} \left[ \frac{x^2 - 2x + 5}{(x-1)^3(x+1)^2} \right]$

$= -\frac{4x}{3} \left[ \frac{(x-1)^2 + 4}{(x-1)^3(x+1)^2} \right]$

Since  $(x-1)^2 + 4 > 0$  for all  $x \in \mathbb{R}$ , we have that  $f'(x) = 0$  if and only if  $x = 0$ . (5)

$x$	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f'(x)$	$< 0$	$< 0$	$> 0$	$< 0$
	$f$ decreases	$f$ decreases	$f$ increases	$f$ decreases

(10)

The function  $f$  has a relative minimum at  $x = 0$ . (5)

Then  $f(0) = 2$ .

When  $x \longrightarrow \pm \infty$  we have that  $f(x) \longrightarrow 0$ .

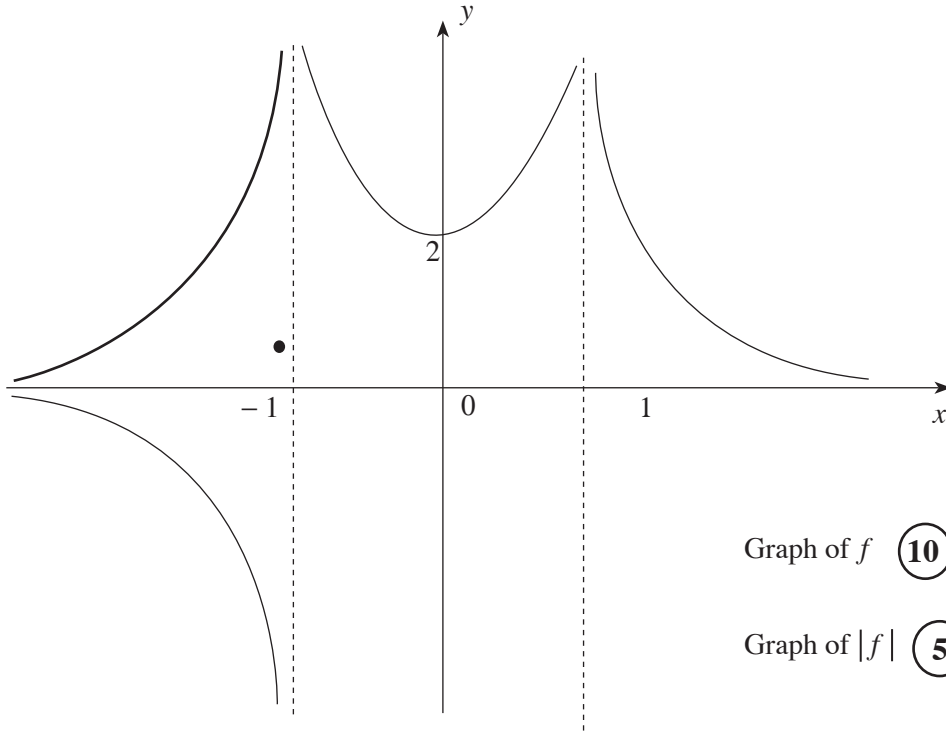
$x \longrightarrow -1^-, f(x) \longrightarrow -\infty$

$x \longrightarrow -1^+, f(x) \longrightarrow +\infty$

$x \longrightarrow 1^-, f(x) \longrightarrow +\infty$

$x \longrightarrow 1^+, f(x) \longrightarrow +\infty$

(10)



Graph of  $f$  (10)

Graph of  $|f|$  (5)

70

15. (a) 
$$I = \int_0^1 \frac{dx}{(2+x)^{1/2} (2-x)^{3/2}} = \int_0^1 \frac{dx}{(4-x^2)^{1/2} (2-x)}$$

By substituting  $x = 2 \sin \theta$ , (5)

$dx = 2 \cos \theta d\theta$  (5)

$x = 0, \sin \theta = 0$

$\theta = 0$

$x = 1, \sin \theta = \frac{1}{2}$   
 $\theta = \frac{\pi}{6}$  (5)

$$I = \int_0^{\pi/6} \frac{2 \cos \theta}{(4 - 4 \sin^2 \theta)^{1/2} (2 - 2 \sin \theta)} d\theta$$
 (5)

$$= \int_0^{\pi/6} \frac{2 \cos \theta}{2 \cos \theta \cdot 2(1 - \sin \theta)} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \frac{1}{2} \int_0^{\pi/6} \sec^2 \theta d\theta + \frac{1}{2} \int_0^{\pi/6} \sec \theta \tan \theta d\theta$$
 (5)

$$= \frac{1}{2} [\tan \theta]_0^{\pi/6} + \frac{1}{2} [\sec \theta]_0^{\pi/6}$$
 (5)

$$= \frac{1}{2} \left[ \frac{1}{3} + \frac{2}{3} - 1 \right] = \frac{3-1}{2} \quad (5)$$

50

(b)  $G(x) = \frac{A}{x+2} + \frac{Bx+C}{x^2+8} \quad (5)$

$$1 = A(x^2+8) + (Bx+C)(x+2) \quad (5)$$

$$\left. \begin{aligned} \text{Coefficient of } x^2 : 0 &= A + B \Rightarrow A = -B \\ \text{Coefficient of } x : 0 &= 2B + C \Rightarrow C = -2B \end{aligned} \right\} \quad (5)$$

$$\text{Constant: } 1 = 8A + 2C \quad (5)$$

$$1 = -8B - 4B \Rightarrow 12B = -1$$

$$\Rightarrow B = -\frac{1}{12} \quad (5)$$

$$A = \frac{1}{12}, \quad C = \frac{1}{6}$$

$$(5) \quad (5)$$

$$g(x) = \int \frac{1}{(x+2)(x^2+8)} dx$$

$$g(x) = \frac{1}{12} \int \frac{1}{x+2} dx - \frac{1}{12} \int \frac{x}{x^2+8} dx + \frac{1}{6} \int \frac{1}{x^2+8} dx \quad (5)$$

$$= \frac{1}{12} \ln|x+2| - \frac{1}{24} \ln(x^2+8) + \frac{1}{6} \tan^{-1} \left( \frac{x}{2\sqrt{2}} \right) + C \quad (5)$$

$$= \frac{1}{24} \ln \left[ \frac{(x+2)^2}{x^2+8} \right] + \frac{1}{6} \frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{x}{2\sqrt{2}} \right) + C \quad (5)$$

$$= \frac{1}{24} \ln \left[ \frac{(x+2)^2}{x^2+8} \right] + \frac{1}{12\sqrt{2}} \tan^{-1} \left( \frac{x}{2\sqrt{2}} \right) + C \quad (5)$$

60

(c)  $I_n = \int x^n \sin x dx$

$$= \int -x^n \frac{d}{dx} (\cos x) \quad (5)$$

$$= -x^n \cos x + \int (\cos x) nx^{n-1} dx \quad (10)$$

$$= -x^n \cos x + n \int x^{n-1} \frac{d}{dx} (\sin x) \quad (5)$$

$$= -x^n \cos x + n \left\{ x^{n-1} \sin x - \int \sin x (n-1) x^{n-2} dx \right\} \quad (10)$$

$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1) I_{n-2} \quad (5)$$

$$I_n + n(n-1) I_{n-2} = x^{n-1} [n \sin x - x \cos x] \quad (5)$$

40

16. (a)

Let  $P(\bar{x}, \bar{y})$  be an arbitrary point on any one of the angle bisectors.

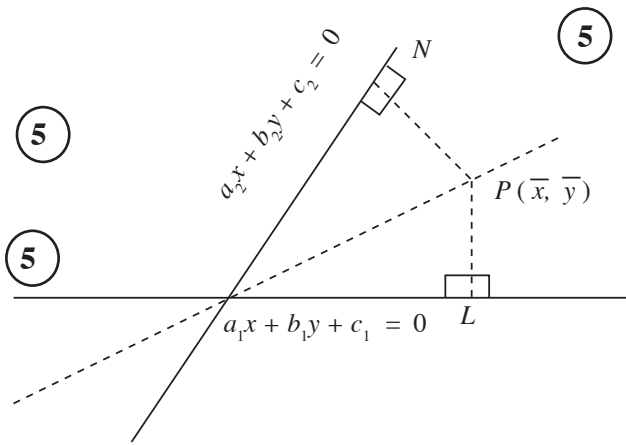
$$PL = PN \quad (5)$$

$$\frac{|a_1\bar{x} + b_1\bar{y} + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2\bar{x} + b_2\bar{y} + c_2|}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$

$$\therefore \frac{a_1\bar{x} + b_1\bar{y} + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2\bar{x} + b_2\bar{y} + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$

By replacing  $\bar{x}$  by  $x$ , and  $\bar{y}$  by  $y$  (5)

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$



The equations of the angle bisectors;

$$\frac{4x + y + 3}{\sqrt{4^2 + 1^2}} = \pm \frac{x + 4y - 3}{\sqrt{1^2 + 4^2}} \quad (5)$$

$$+ : 3x - 3y + 6 = 0 \Rightarrow x - y + 2 = 0 \quad (5)$$

$$- : 5x + 5y = 0 \Rightarrow x + y = 0$$

By solving  $x + y = 0$  and  $x - y + 2 = 0$  we obtain

$$x = -1, \quad y = 1 \quad (5)$$

Let  $A = (-1, 1)$

$B = (0, 2)$  lies on the line given by  $x - y + 2 = 0$ . (5)

Let  $P = (x, y)$  be a point on the line given by  $x + y = 0$

Since  $PA$  is perpendicular to  $PB$ ,



$$\left(\frac{y-1}{x+1}\right) \times 1 = -1 \quad (5)$$

$$\frac{y-1}{-1} = \frac{x+1}{1} = t; \quad t \text{ is a parameter.} \quad (5)$$

$$\therefore x = -1 + t, \quad y = 1 - t$$

Let  $T$  be the value of  $t$  corresponding to the point  $D$  which lies on  $x + y = 0$  and is such that  $AD = AB$ .

$$\text{Then } D = (-1 + T, 1 - T) \quad (5)$$

$$AD^2 = AB^2 \Rightarrow T^2 + T^2 = 1^2 + 1^2 = 2 \quad (5)$$

$$T = \pm 1 \quad (5)$$

$$\therefore D = (0, 0) \text{ or } (-2, +2) \quad (5) \quad (5)$$

When  $D \equiv (0, 0)$  the equation of the side  $CD$  is

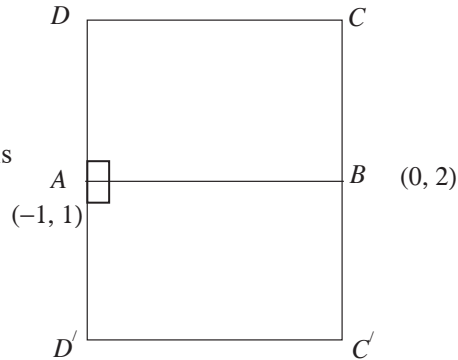
$$x - y = 0 \quad (5)$$

When  $D' \equiv (-2, +2)$  the equation of the side  $C'D'$  is

$$x - y + 4 = 0 \quad (5)$$

The equation of the side  $BC$  and the side  $BC'$  is

$$x + y - 2 = 0 \quad (5)$$



100

(b)  $S^1 = x^2 + y^2 - 2x + 4y - 3 = 0$

Let  $S = x^2 + y^2 + 2gx + 2fy + c = 0$ . Here  $g, f, c$  are constants. (5)

Since  $S = 0$  is bisected by  $S^1 = 0$ , the centre of  $S = 0$  lies on the line (5)

$S^1 - S = 0$  given by

$$-2x(g+1) - 2y(f-2) - 3 - c = 0 \quad (5)$$

$$\therefore 2(g)(g+1) + 2(f)(f-2) - c - 3 = 0 \quad (1) \quad (5)$$

Since the circle  $S = 0$  passes through the point  $(1, 1)$ ,

$$1^2 + 1^2 + 2g + 2f + c = 0$$

$$\therefore c = -2g - 2f - 2 \quad (2) \quad (5)$$

From (1) and (2),

$$2g^2 + 2g + 2f^2 - 4f - (-2g - 2f - 2) - 3 = 0 \quad (5)$$

$$2g^2 + 2f^2 + 4g - 2f - 1 = 0 \quad (5)$$

$$2(-g)^2 + 2(-f)^2 - 4(-g) + 2(-f) - 1 = 0$$

$$\therefore \text{the point } (-g, -f) \text{ lies on the circle } 2x^2 + 2y^2 - 4x + 2y - 1 = 0 \quad (5)$$

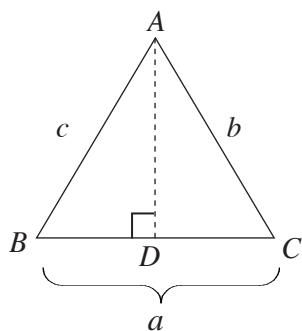
The centre of this circle is  $(1, -\frac{1}{2})$  and its radius is  $(5)$

$$r = \sqrt{1^2 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{4}}$$

$$= \frac{\sqrt{7}}{2} \quad (5)$$

50

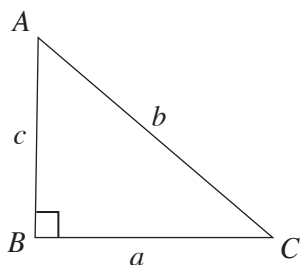
17. (a)



$$a = BC = BD + DC$$

$$a = c \cos B + b \cos C$$

(5)

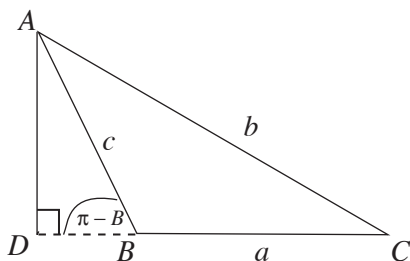


$$a = b \cos C + 0$$

$$= b \cos C + c \cos 90^\circ$$

$$= b \cos C + c \cos B$$

(5)



$$a = BC = CD - BD$$

$$= b \cos C - c \cos (\pi - B)$$

$$= b \cos C + c \cos B$$

(5)

Similarly,  $b = a \cos C + c \cos A$

$$a \cos C = b - c \cos A$$

$$a^2 \cos^2 C = b^2 - 2bc \cos A + c^2 \cos^2 A$$

$$a^2 - a^2 \sin^2 C = b^2 + c^2 - 2bc \cos A - c^2 \sin^2 A$$

(10)

$$\underbrace{a^2 + c^2 \sin^2 A - a^2 \sin^2 C}_{= 0} = b^2 + c^2 - 2bc \cos A ; \text{ since } \frac{a}{\sin A} = \frac{c}{\sin C} \quad (5)$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$a, b, c$  are in an arithmetic progression. Therefore,

$$a + c = 2b \quad (5)$$

$$b \cos C + c \cos B + a \cos B + b \cos A = 2b \quad (5)$$

$$\cos A + \cos C + 2 \cos B = 2$$

$$2 \cos \left( \frac{A+C}{2} \right) \cos \left( \frac{A-C}{2} \right) = 2(1 - \cos B)$$

$$(5)$$

$$2 \cos \left( \frac{\pi}{2} - \frac{B}{2} \right) \cos \left( \frac{A-C}{2} \right) = 4 \sin^2 \frac{B}{2}$$

$$\cos \left( \frac{A-C}{2} \right) = 2 \sin \frac{B}{2} \quad (5)$$

50

(b)  $0 < x, y < \frac{\pi}{2}$

$$\therefore 0 < \frac{\pi}{2} - y < \frac{\pi}{2} \quad (5)$$

$$\sin x > \cos y = \sin \left( \frac{\pi}{2} - y \right) \quad (5)$$

$$\sin x > \sin \left( \frac{\pi}{2} - y \right)$$

As the angle increases in the domain  $(0, \frac{\pi}{2})$ , the sine value also increases.

Therefore, (10)

$$\therefore x > \frac{\pi}{2} - y \quad (5)$$

$$x + y > \frac{\pi}{2}$$

25

(c)  $f(x) = 3 \cos^2 x + 8 \sin x \cos x - 3 \sin^2 x$

$$= 3 \cos 2x + 4 \sin 2x \quad (5)$$

$$(5)$$

$$= 5 \left( \frac{3}{5} \cos 2x + \frac{4}{5} \sin 2x \right) \quad (5)$$

$$= 5(\sin \alpha \cos 2x + \cos \alpha \sin 2x)$$

$$= 5 \sin(2x + \alpha)$$

$$= A \sin(2x + \alpha) \quad (5)$$

Here  $A = 5$ , and  $\alpha$  is an acute angle such that  $\tan \alpha = \frac{3}{4}$ . (5)

$$f(x) = \frac{5}{2}$$

$$5 \sin(2x + \alpha) = \frac{5}{2}$$

$$\sin(2x + \alpha) = \frac{1}{2} = \sin \frac{\pi}{6} \quad (5)$$

$$2x + \alpha = n\pi + (-1)^n \frac{\pi}{6} \quad (5)$$

$$x = \frac{n\pi}{2} - \frac{\alpha}{2} + (-1)^n \frac{\pi}{12}, \text{ here } n \in Z$$

(5)

$$f(x) = 5 \sin(2x + \alpha)$$

$$\text{Maximum } f(x) = 5; x = \frac{\pi}{4} - \frac{\alpha}{2}$$

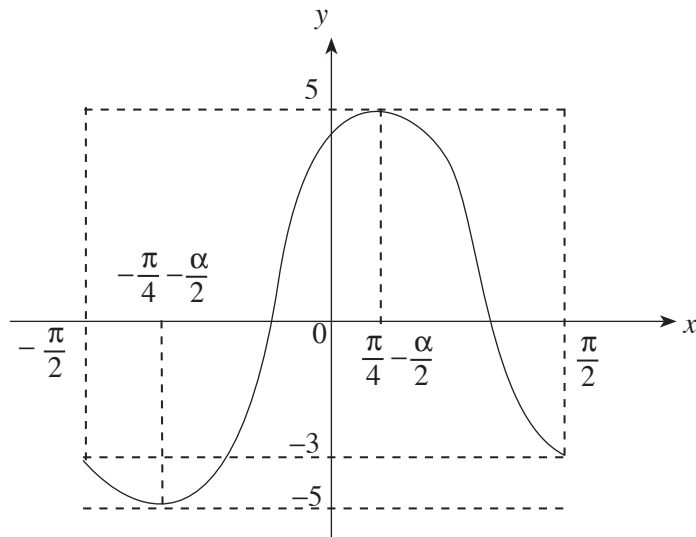
(5)

(5)

$$\text{Minimum } f(x) = -5; x = -\frac{\pi}{4} - \frac{\alpha}{2} \quad (\text{Since } \alpha < \frac{\pi}{4})$$

(5)

(5)



(15)

**G.C.E. (A.L.) Support Seminar - 2016**  
**Combined Mathematics - Paper II**  
**Answer Guide**

**Part A**

1. Applying  $v^2 = u^2 + 2as$  to  $m \downarrow$

$$v^2 = 2gh$$

$$\therefore v = \sqrt{2gh}$$

(5)

Applying  $I = \Delta(mv)$ ;

To  $P$  and  $m \downarrow$

$$-J = (2m + m)v_1 - mv - 2m \times 0$$

(5)

$$\therefore -J = 3mv_1 - mv \quad \text{--- (1)}$$

To  $Q \uparrow$

$$J = 2mv_1 - 0 \quad \text{--- (2)}$$

(5)

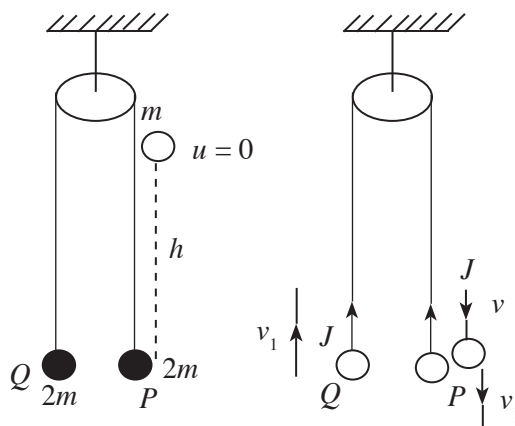
From (1) and (2),

$$v_1 = \frac{v}{5} = \frac{\sqrt{2gh}}{5}$$

(5)

$$J = \frac{2m}{5} \sqrt{2gh}$$

(5)



25

2. The volume of water that is ejected in a second =  $8 (0.005) \text{ m}^3$

$$= 0.040 \text{ m}^3$$

(5)

The mass of water that is ejected in a second =  $10^3 \times 0.040 \text{ kg}$

$$= 40 \text{ kg}$$

(5)

Work done by the pump in a second

$$= mgh + \frac{1}{2} mv^2$$

$$= (40 \times 10 \times 4) + \frac{1}{2} \times 40 \times 8^2$$

(5)

(5)

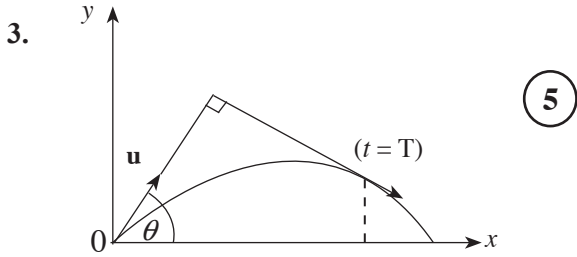
$$= 2880 \text{ js}^{-1}$$

$\therefore$  the power of the pump

$$= 2880 \text{ W}$$

(5)

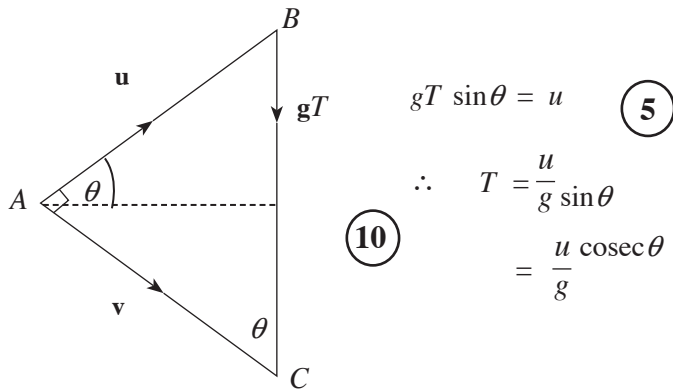
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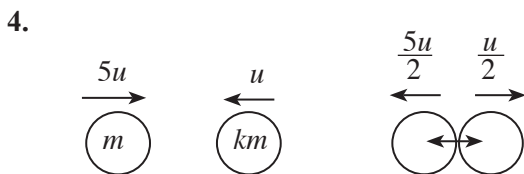
When  $t = T$

$$\mathbf{v} = \mathbf{u} + \mathbf{g}T \quad (5)$$

$$\vec{AC} = \vec{AB} + \vec{BC}$$



25



By apply the law of conservation of momentum to the system

$$\longrightarrow 5mu - kmu = \frac{km u}{2} - \frac{5mu}{2} \quad (5)$$

$$10 - 2k = k - 5$$

$$\therefore k = 5 \quad (5)$$

From Newton's law of restitution

$$\frac{u}{2} + \frac{5u}{2} = e(u + 5u) \quad (5)$$

$$3u = 6ue$$

$$\frac{1}{2} = e \quad (5)$$

$$I = \Delta(mv)$$

$$\longrightarrow -I = -m \cdot \frac{5u}{2} - m \cdot 5u$$

$$I = \frac{15mu}{2} \quad (5)$$

25

5. Since  $\underline{a} \perp \underline{b}$ , we have  $\underline{a} \cdot \underline{b} = 0$  (5)

$$\therefore (2\mathbf{i} + 3\mathbf{j}) \cdot (\lambda\mathbf{i} + \mu\mathbf{j}) = 0$$

$$2\lambda + 3\mu = 0 \quad \text{--- (1) (5)}$$

Since  $|\underline{b}| = 1$ , we have  $\lambda^2 + \mu^2 = 1$  --- (2) (5)

From (1) and (2),  $\mu = \pm\sqrt{\frac{2}{13}}$

Since  $\mu > 0$ ,  $\mu = \sqrt{\frac{2}{13}}$  (5)

From (1),  $\lambda = -\sqrt{\frac{3}{13}}$  (5)

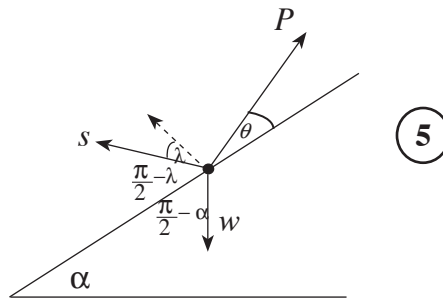
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6. The object is in limiting equilibrium.

By Lami's Theorem,

$$\frac{P}{\sin[\pi - (\alpha + \lambda)]} = \frac{w}{\sin[\frac{\pi}{2} - (\theta - \lambda)]} \quad (5)$$

$$P = \frac{w \sin(\lambda + \alpha)}{\cos(\theta - \lambda)} \quad (5)$$



For  $P$  to be minimum,  $(\theta - \lambda)$  should be maximum.

That is,  $\theta = \lambda$  (5)

$$\therefore P (\text{minimum}) = w \sin(\lambda + \alpha) \quad (5)$$

25

7. Let  $P(A) = \frac{1}{3}$  and  $P(B) = \frac{1}{4}$ .

(A)  $\longrightarrow$  1<sup>st</sup> (First)    (B)  $\longrightarrow$  2<sup>nd</sup> (Second)

(i)  $X = (A \cap B') \cup (A' \cap B)$  (5)

But  $(A \cap B') \cap (A' \cap B) = \phi$

$$\therefore P(X) = P(A \cap B') + P(A' \cap B) \quad (\because \text{By Axiom III})$$

$$= P(A) P(B') + P(A') P(B) \quad (5) \quad (\text{Since the events are independent})$$

$$= \frac{1}{3} \times \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4}$$

$$= \frac{1}{3} + \frac{2}{3} \times \frac{1}{4} = \left(\frac{1}{3} + \frac{2}{12}\right) = \frac{5}{12} \quad (5)$$

(ii)  $P(A|X) = \frac{P(A \cap X)}{P(X)} = \frac{P(A) P(B')}{P(X)} \quad (5)$

$$= \frac{\frac{1}{3} \times \frac{3}{4}}{\frac{5}{12}} = \frac{3}{5} \quad (5)$$

25

$$\begin{aligned} 8. \quad P(A \cap B') &= 0.2, \quad P(A' \cap B) &= 0.1 & \textcircled{5} \\ P(A' \cap B') &= P(A \cup B)' &= 0.6 \\ 1 - P(A \cup B) & &= 0.6 \\ P(A \cup B) & &= 0.4 & \textcircled{5} \\ P(A \cup B) - P(A \cap B) & &= 0.2 + 0.1 \\ \therefore P(A \cap B) & &= 0.4 - 0.3 = 0.1 & \textcircled{5} \\ P(A' \cap B) & &= P(B) - P(A \cap B) \\ 0.1 + 0.1 & &= P(B) & \textcircled{5} \\ \therefore P(A|B) & &= \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.2} & \textcircled{5} \\ & &= \frac{1}{2} & \textcircled{5} \end{aligned}$$

25

$$\begin{aligned} 9. \quad \bar{x} &= 5 \text{ and } s_x = 2 \\ \text{(i) } y_i &\in \{12, 13, 14, 15, 16, 17, 18\} \\ \text{Let } y_i &= x_i + 10. \\ \text{Here } x_i &\in \{2, 3, 4, 5, 6, 7, 8\} \\ \therefore \bar{y} &= \bar{x} + 10 = 5 + 10 = 15 & \textcircled{5} \\ \text{and } s_y &= s_x = 2 \end{aligned}$$

$$\begin{aligned} \text{(ii) } y_i &\in \{20, 30, 40, 50, 60, 70, 80\} \\ \text{Let } y_i &= 10x_i \\ \text{Here } x_i &\in \{2, 3, 4, 5, 6, 7, 8\} \\ \therefore \bar{y} &= 10\bar{x} & \textcircled{5} \\ &= 10 \times 5 = 50 \\ \text{and } s_y &= 10s_x = 10 \times 2 = 20 & \textcircled{5} \end{aligned}$$

$$\begin{aligned} \text{(iii) Let } y_i &= ax_i + b. & \textcircled{5} \\ \text{Then } \bar{y} &= a\bar{x} + b = 5a + b \\ s_y^2 &= a^2 s_x^2 \\ s_y &= |a| s_x \\ &= 2|a| & \textcircled{5} \end{aligned}$$

25



10.

$u_i$	-3	-2	-1	0	1	2
$f_i$	5	10	25	30	20	10
$f_i u_i$	-15	-20	-25	0	20	20

(5)

$$\bar{u} = \frac{\sum f_i u_i}{\sum f_i} = -\frac{20}{100} = -\frac{1}{5}$$

$$u_i = \frac{x_i - 35}{a}$$

(5)

$$\therefore \bar{x} = a\bar{u} + 35$$

$$33 = -\frac{a}{5} + 35$$

(5)

$$a = 10$$

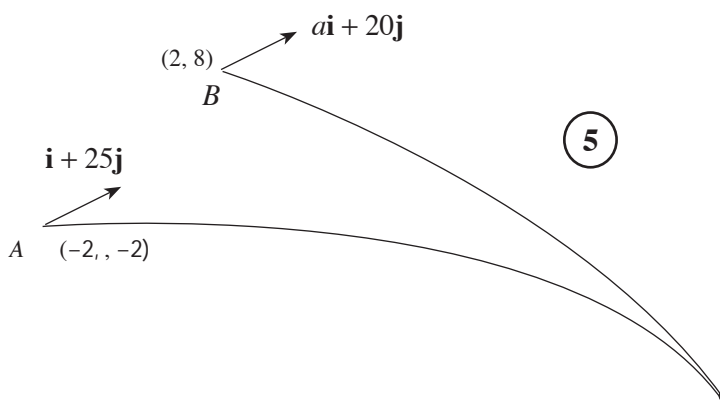
(5)

Intervals	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
$f_i$	5	10	25	30	20	10

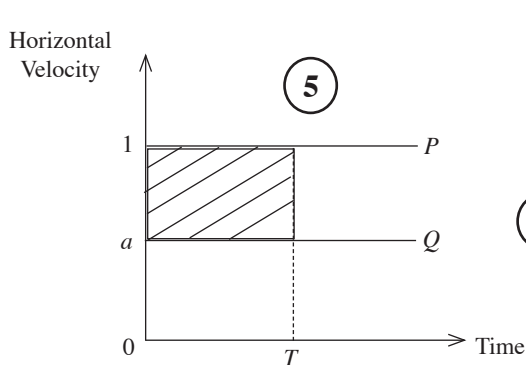
(5)

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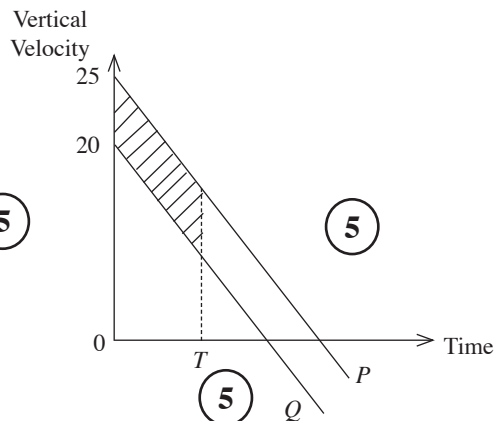
11. (a)



(5)



(5)



(5)

To collide,

$$\text{Vertical displacement of } P = \text{Vertical displacement of } Q + 10$$

$$\text{Vertical displacement of } P - \text{Vertical displacement of } Q = 10$$

(10)

$$5T = 10$$

$$T = 2 \quad (5)$$

Horizontal displacement of  $P =$  Horizontal displacement of  $Q + 4$

$$\text{Horizontal displacement of } P - \text{Horizontal displacement } Q = 4 \quad (10)$$

$$(1 - a) 2 = 4 \quad (5)$$

$$1 - a = 2 \quad (5)$$

$$a = -1 \quad (5)$$


60

(b)  $v(S, E) = \longrightarrow u$


$v(P, S) =$    $v \quad (5)$

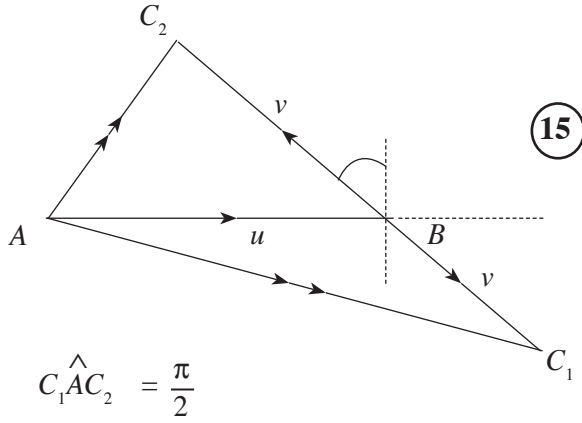
$$v(P, E) = v(P, S) + v(S, E)$$

Out

$$\underline{v}(P, E) =$$
   $\Rightarrow \vec{BC}_1 + \vec{AB} = \vec{AC}_1 \quad (5)$

In

$$\underline{v}(P, E) =$$
   $\Rightarrow \vec{BC}_2 + \vec{AB} = \vec{AC}_2 \quad (5)$



Hence, the circle with diameter  $C_1C_2$  passes through the point  $A$ .

Furthermore, since the midpoint of  $C_1C_2$  is  $B$ , (10)

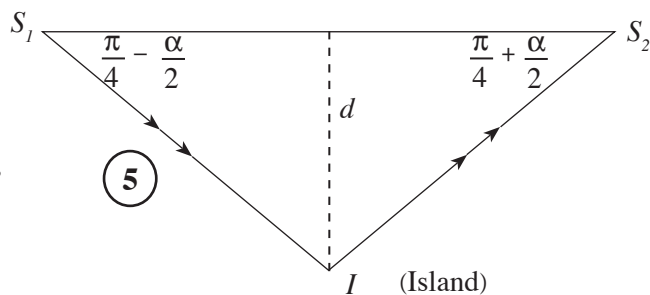
$$BC_1 = BC_2 = BA = u \quad (5)$$

$$v = u$$

$$\widehat{BAC_1} = \frac{\pi}{4} - \frac{\alpha}{2} \quad \text{and} \quad \widehat{BAC_2} = \frac{\pi}{4} + \frac{\alpha}{2}$$

$$(5)$$

$$(5)$$



If the total time for the journey is  $t$ ,

$$\begin{aligned}
 t &= \frac{S_1 I}{AC_1} + \frac{S_2 I}{AC_2} && \text{(5)} \\
 &= \frac{S_1 I \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{AC_1 \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} + \frac{S_2 I \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)}{AC_2 \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)} && \text{(10)} \\
 &= \frac{d}{v \cos \alpha} + \frac{d}{v \cos \alpha} = \frac{2d}{u \cos \alpha} \quad (\because v = u) && \text{(5)}
 \end{aligned}$$

12. (a) By the law of conservation of energy;

$$\frac{1}{2}mu^2 - mga = mga \cos \theta + \frac{1}{2}mv^2 \quad \text{(15)}$$

$$v^2 - u^2 + 2ga(1 + \cos \theta) = 0 \quad \text{(1)} \quad \text{(5)}$$

Applying  $F = ma$  to  $m$

$$R + mg \cos \theta = \frac{mv^2}{a} \quad \text{(2)} \quad \text{(10)}$$

By substituting from (1) into (2)

$$R + mg \cos \theta = \frac{m}{a} [u^2 - 2ga(1 + \cos \theta)] \quad \text{(5)}$$

$$R = \frac{mu^2}{a} - mg(2 + 3 \cos \theta)$$

If the particle leaves the surface when  $OA$  makes an angle of  $\alpha$  with the upward vertical, then at that point  $R = 0$ . (5)

$$\therefore u^2 - 2ga - 3ga \cos \alpha = 0 \quad \text{(5)}$$

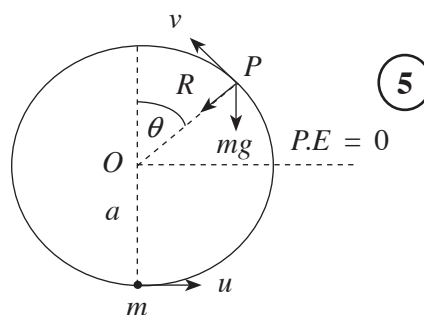
$$\cos \alpha = \frac{u^2 - 2ga}{3ga} > 0 \quad (\because u^2 > 2ga)$$

$\therefore \alpha$  is an acute angle.

Furthermore, since  $0 < \alpha < \frac{\pi}{2}$ , we have that  $0 < \cos \alpha < 1$ .

$$\frac{u^2 - 2ga}{3ga} < 1 \quad \text{(5)}$$

$$u^2 < 5ga$$



(5)

P.E = 0

(5)

When the particle of mass  $m$  leaves the surface,  $\cos \alpha = \frac{1}{3}$ ,

$$\frac{1}{3} = \frac{u^2 - 2ga}{3ga}$$

$$u^2 - 2ga = \frac{2}{3} ga$$

$$u^2 = (2 + \frac{2}{3}) ga \quad (5)$$

Then the velocity  $v^2 = u^2 - 2ga (1 + \frac{1}{3}) = 2ga + \frac{2}{3} ga - 2ga - \frac{2ga}{3} = \frac{ga}{3} \quad (5)$

After the particle of mass  $m$  leaves the surface of the sphere, its motion is that of a projectile.

In the ensuing motion, if the time taken to travel a horizontal distance  $a \sin \alpha$  is  $t_0$ '

$$a \sin \alpha = (v \cos \alpha) t_0 \quad (5)$$

Then the distance traveled upward  $y = (v \sin \alpha) t_0 - \frac{1}{2} g t_0^2$

$$y = \frac{v \sin \alpha \times a \sin \alpha}{v \cos \alpha} - \frac{1}{2} \frac{ga^2 \sin^2 \alpha}{v^2 \cos^2 \alpha} \quad (5)$$

$$= \frac{\frac{2}{3} a}{\frac{1}{3}} - \frac{ga^2}{\frac{2ga}{3}} \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$= \frac{2a}{3} - \frac{3}{3} a$$

$$= -\frac{a}{3}$$

$$= -a \cos \alpha \quad (5)$$

Since the particle of mass  $m$  has travelled a distance of  $a \cos \alpha$  downward when it passes the vertical line through  $O$ , it passes through the centre  $O$  of the sphere.

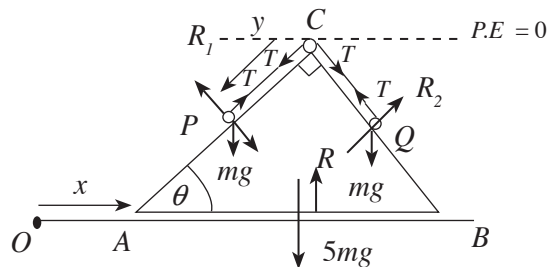
80

(b)

$$\mathbf{v}(P, O) = \begin{matrix} \dot{y} \\ \theta \\ \dot{x} \end{matrix}$$

$$\mathbf{v}(Q, O) = \begin{matrix} +\dot{y} \\ \pi - \theta \\ -2 \\ \dot{x} \end{matrix}$$

(5)



Applying the law of conservation of momentum;

$$\longrightarrow 5m \dot{x} + m(\dot{x} - \dot{y} \cos \theta) + m(\dot{x} - \dot{y} \sin \theta) = 0 \quad (10)$$

$$7m \dot{x} = m \dot{y} (\cos \theta + \sin \theta)$$

$$7\dot{x} = \dot{y} \left( \frac{3}{5} + \frac{4}{5} \right) \quad (5)$$

$$5\dot{x} = \dot{y} \quad \text{—————} \quad (1)$$

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Applying the law of conservation of energy;

$$\frac{1}{2} 5m \dot{x}^2 + \frac{1}{2} m \{(\dot{x} - \dot{y} \cos \theta)^2 + (\dot{y} \sin \theta)^2\} \quad (20)$$

$$+ \frac{1}{2} m \{(\dot{x} - \dot{y} \sin \theta)^2 + (\dot{y} \cos \theta)^2\} - mgy \sin \theta - mg(l - y) \cos \theta = \text{constant}$$

$$5\dot{x}^2 + \{ \dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y} \cos \theta \} + \{ \dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y} \sin \theta \}$$

$$- 2gy \sin \theta + 2gy \cos \theta = \text{constant} \quad (5)$$

$$7\dot{x}^2 + 2\dot{y}^2 - 2\dot{x}\dot{y} \left( \frac{4}{5} + \frac{3}{5} \right) - 2gy \frac{4}{5} + 2gy \frac{3}{5} = \text{constant}$$

$$35\dot{x}^2 + 10\dot{y}^2 - 14\dot{x}\dot{y} - 2gy = \text{constant} \quad \text{—————} \quad (2)$$

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From (1) and (2)

$$35\dot{x}^2 + 250\dot{x}^2 - 70\dot{x}^2 - 2gy = \text{constant}$$

$$215\dot{x}^2 - 2gy = \text{constant}$$

Differentiating with respect to  $t$ ,


$$430\dot{x} \cdot \ddot{x} - 2g\dot{y} = 0 \quad (5)$$

$$430\dot{x} \cdot \ddot{x} - 2g \cdot 5\dot{x} = 0 \quad (\because \dot{x} \neq 0) \quad (5)$$

$$\therefore \ddot{x} = \frac{g}{43}$$

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Applying  $F = ma$  to  $P$



$$: mg \sin \theta - T = m(\ddot{y} - \ddot{x} \cos \theta) \quad (5)$$

$$T = mg \sin \theta - m(5\ddot{x} - \ddot{x} \cos \theta) \quad (5)$$

$$= mg \frac{4}{5} - m\ddot{x} \left( 5 - \frac{3}{5} \right)$$

$$= \frac{4mg}{5} - m \cdot \frac{1}{43} g \cdot \frac{22}{5}$$

$$= \frac{2mg}{5} \left\{ 2 - \frac{11}{43} \right\}$$

$$= \frac{2mg}{5} \times \frac{75}{43}$$

$$= \frac{30mg}{43} \quad (5)$$

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13. If the tension in the string when it is extended a length  $x$  from its natural length is  $T$ , then

$$T = \frac{\lambda x}{a} = \frac{2mgx}{a} \quad (5)$$

Applying  $F = ma$  to the motion of the particle;

$$mg \sin 30^\circ - T = m\ddot{x} \quad (10)$$

$$mg \times \frac{1}{2} - \frac{2\lambda gx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a} \left(x - \frac{a}{4}\right) \quad (1) \quad (5)$$

$$x = \frac{a}{4} + A \cot \omega t + B \sin \omega t \quad (2)$$

$$\dot{x} = -A\omega \sin \omega t + B\omega \cos \omega t \quad (3) \quad (5)$$

$$\ddot{x} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \quad (4) \quad (5)$$

$$= -\omega^2 (A \cos \omega t + B \sin \omega t)$$

$$\ddot{x} = -\omega^2 \left(x - \frac{a}{4}\right) \quad (5) \text{ by } (2) \quad (5)$$

By considering (1) and (5)  $\omega^2 = \frac{2g}{a}$ . (5)

$$t = 0 \text{ when } \dot{x} = 0. \quad (5) \quad \omega = \sqrt{\frac{2g}{a}}$$

From (3),  $0 = B\omega$

Since  $\omega \neq 0$  we have that  $B = 0$ . (5)

$x = a$  when  $t = 0$  (5)

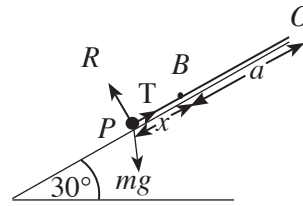
From (2),  $a - \frac{a}{4} = A \Rightarrow A = \frac{3a}{4}$  (5)

$$\therefore x = \frac{3a}{4} \cos \omega t + \frac{a}{4}$$

$$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t$$

The centre of the oscillation is given by  $x - \frac{a}{4} = 0$ . (5)

That is,  $x = \frac{a}{4}$  is the centre. (5)



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When  $\dot{x} = 0$  at the amplitude, let  $t = t_1$ .

$$0 = -A\omega \sin \omega t_1 \quad (5)$$

$$\sin \omega t_1 = 0$$

$$\omega t_1 = n\pi ; n \in \mathbb{Z}_0^+ \quad (5)$$

$$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t_1$$

$$x - \frac{a}{4} = \pm \frac{3a}{4} \quad (5)$$

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$$\therefore \text{the amplitude of the simple harmonic motion of the particle} = \frac{3a}{4} \quad (5)$$

Let the velocity of the particle when it first arrives at the natural length of the string be  $V$ .

$$\text{Then } x = 0. \quad (5)$$

$$\frac{3a}{4} \cos \omega t = -\frac{a}{4}$$

$$\cos \omega t = -\frac{1}{3} \quad (5)$$

$$\begin{aligned} V &= -A\omega \sin \omega t \\ &= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \sqrt{1 - \cos^2 \omega t} \\ &= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \sqrt{\frac{8}{9}} \quad (5) = -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \frac{2\sqrt{2}}{3} \end{aligned}$$

$$= -\sqrt{ag}$$

$$\text{The velocity of the particle when } x = 0 \text{ is } \uparrow \sqrt{ag}. \quad (5)$$

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Let  $t_0$  be the time when the particle first arrives at the natural length of the string.

$$\text{Then } x = 0. \quad (5)$$

$$-\frac{a}{4} = \frac{3a}{4} \cos \omega t_0$$

$$\cos \omega t_0 = -\frac{1}{3} \quad (5)$$

$$\omega t_0 = \pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$t_0 = \frac{1}{\omega} \left[ \pi - \cos^{-1}\left(\frac{1}{3}\right) \right] = \sqrt{\frac{a}{2g}} \left[ \pi - \cos^{-1}\left(\frac{1}{3}\right) \right] \quad (5)$$

The particle travels up to the point  $O$  under gravity.

If the time taken for the particle to travel from  $B$  to  $O$  is  $t_2$ ,

$$S = ut + \frac{1}{2} at^2 \quad \uparrow$$

$$S = a, u = \sqrt{ag}, a = -g \sin 30^\circ$$

$$a = \sqrt{ag}t_2 - \frac{1}{2} \frac{g}{2} t_2^2 \quad (5)$$

$$\frac{g}{4} t_2^2 - \sqrt{ag} t_2 + a = 0$$

$$t_2 = \frac{\sqrt{ag} \pm \sqrt{ag - 4 \frac{g}{4} a}}{\frac{g}{2}}$$

$$t_2 = 2\sqrt{\frac{a}{g}} \quad (5)$$

$\therefore$  the time taken to travel to  $O = t_0 + t_2$ ,

$$= \sqrt{\frac{a}{2g}} (\pi - \cos^{-1}(\frac{1}{3})) + 2\sqrt{\frac{a}{g}}$$

(5)

$$= \sqrt{\frac{a}{2g}} [\pi - \cos^{-1}(\frac{1}{3}) + 2\sqrt{2}]$$

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The string is at its maximum length at  $A$ ; that is when  $x = a$ ,

$$T_A = \frac{\lambda a}{a} \quad (5)$$

$$= \lambda$$

$$T_A = 2mg \quad (5)$$

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14. (a) (i)  $|\underline{a}| = |\underline{b}| = |\underline{c}| = 1 \quad (5)$

If  $(\underline{a} + 2\underline{b}) \perp (5\underline{a} - 4\underline{b})$ ,

$$(\underline{a} + 2\underline{b}) \cdot (5\underline{a} - 4\underline{b}) = 0 \quad (5)$$

$$5\underline{a} \cdot \underline{a} + 10\underline{b} \cdot \underline{a} - 4\underline{a} \cdot \underline{b} - 8\underline{b} \cdot \underline{b} = 0$$

$$5|\underline{a}|^2 + 10\underline{a} \cdot \underline{b} - 4\underline{a} \cdot \underline{b} - 8|\underline{b}|^2 = 0$$

$$5 + 6\underline{a} \cdot \underline{b} - 8 = 0$$

$$6\underline{a} \cdot \underline{b} = 3 \quad (5)$$

$$\underline{a} \cdot \underline{b} = \frac{1}{2}$$

$$|\underline{a}| |\underline{b}| \cos \theta = \frac{1}{2} \quad (5)$$

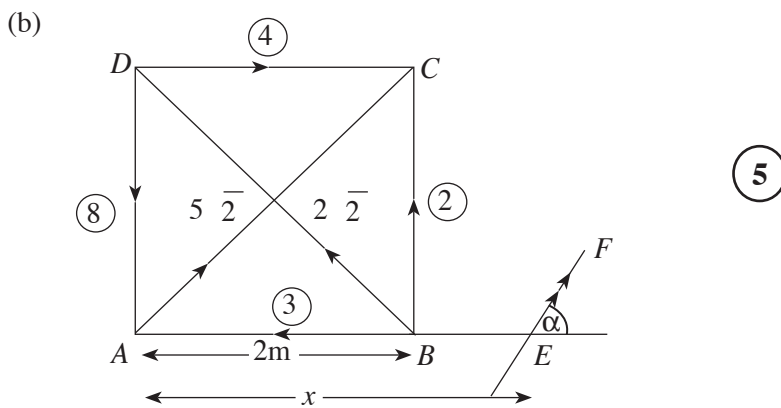
$$1 \times 1 \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ \quad (5)$$

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(ii)  $|a - b|^2 + |b - c|^2 + |c - a|^2$   
 $= (\underline{a} - \underline{b}) \cdot (\underline{a} - \underline{b}) + (\underline{b} - \underline{c}) \cdot (\underline{b} - \underline{c}) + (\underline{c} - \underline{a}) \cdot (\underline{c} - \underline{a})$  (5)  
 $= |a|^2 + |b|^2 - 2a \cdot b + |b|^2 + |c|^2 - 2b \cdot c + |c|^2 + |a|^2 - 2c \cdot a$  (5)  
 $= 6 - 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a})$  (5)  
 $\therefore 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a}) = 6 - (|a - b|^2 + |b - c|^2 + |c - a|^2)$  (5) (1)  
 $|a + b + c|^2 \geq 0$  (5)  
 $\therefore (\underline{a} + \underline{b} + \underline{c}) \cdot (\underline{a} + \underline{b} + \underline{c}) \geq 0$  (5)  
 $|a|^2 + \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{a} + |b|^2 + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{b} + |c|^2 \geq 0$  (5)  
 $3 + 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a}) \geq 0$  (2) (5)  
 From (1) and (2)  
 $3 + 6 - (|a - b|^2 + |b - c|^2 + |c - a|^2) \geq 0$  (5)  
 $\therefore |a - b|^2 + |b - c|^2 + |c - a|^2 \leq 9$  (5)

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(i)  $\vec{X} = 4 - 3 + 5\sqrt{2} \cos 45^\circ - 2\sqrt{2} \cos 45^\circ$  (5)  
 $= 4\text{N}$   
 $\uparrow Y = 2 - 8 + 5\sqrt{2} \cos 45^\circ + 2\sqrt{2} \cos 45^\circ$  (5)  
 $= 1\text{N}$   
 If the resultant is  $R$ ,  
 $R = \sqrt{X^2 + Y^2} = \sqrt{4^2 + 1^2}$   
 $= \sqrt{17}\text{N}$  (5)  
 If the resultant makes an angle  $\alpha$  with the horizontal,  
 $\tan \alpha = \frac{1}{4}$   
 $\alpha = \tan^{-1}\left(\frac{1}{4}\right)$  (5)

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If the point at which the line of action of the resultant intersects  $AB$  is  $E$ , let  $AE = x$ .

$$\overset{\curvearrowleft}{A} \quad 1 \times x = 2 \times 2 - 4 \times 2 + 2 \sqrt{2} \cdot 2 \cos 45^\circ \quad (5)$$

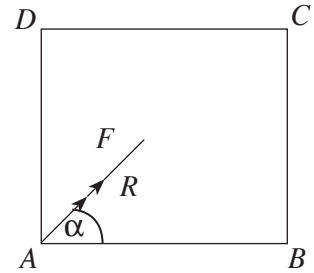
$$x = 0 \quad (5)$$

$A \equiv E$  (Coincides)

The line of action of the resultant passes through  $A$ . (5)

$\therefore$  for the system of forces to be in equilibrium, a force of  $\sqrt{17} \text{ N}$

should be introduced at  $A$  in the direction  $\vec{FA}$ . (5)



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(ii) To reduce the system to a couple of magnitude  $39 \text{ Nm}$  acting in the sense  $ABC$ ,

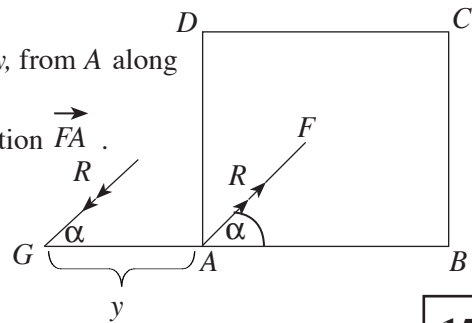
a force of  $\sqrt{17} \text{ N}$  should be introduced at a distance, say  $y$ , from  $A$  along

$BA$  produced, such that it is parallel to  $AF$  and in the direction  $\vec{FA}$ .

$$\overset{\curvearrowleft}{ABC} \quad \sqrt{17} \times AG \sin \alpha = 39 \quad (5)$$

$$\sqrt{17} \times AG \times \frac{1}{\sqrt{17}} = 39$$

$$AG = 39 \text{ m} \quad (5)$$



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(iii) The couple that should be introduced to reduce the system to

a single force acting at  $B$ ;

$$= \curvearrowleft \quad \sqrt{17} \times BA \sin \alpha \quad (5)$$

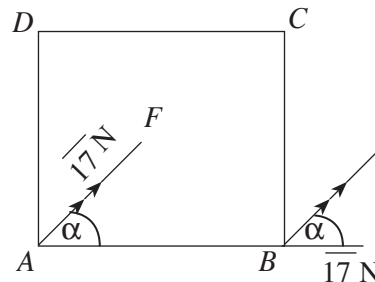
$$= \curvearrowleft \quad \sqrt{17} \times 2 \times \frac{1}{\sqrt{17}} = 2 \text{ Nm} \quad (5)$$

Aliter

If the couple that should be introduced is  $M$ ,

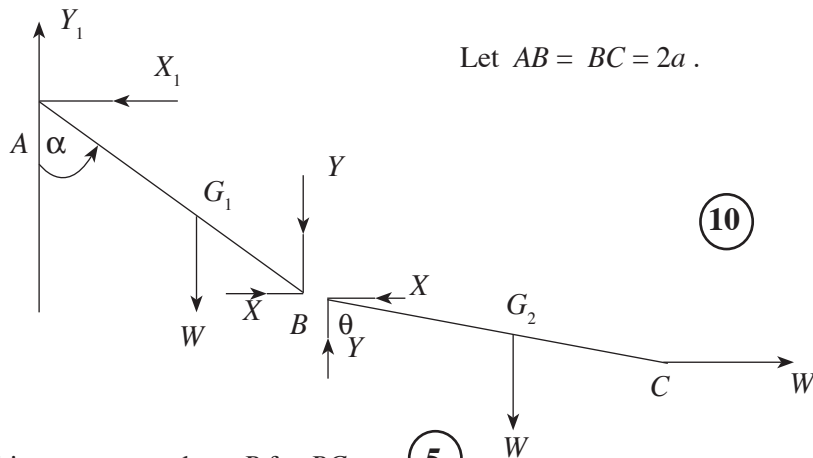
$$M - \sqrt{17} \times 2 \sin \alpha = 0$$

$$M = \sqrt{17} \times 2 \times \frac{1}{\sqrt{17}} = 2 \text{ Nm}$$



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15. (a)



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(i) Taking moments about  $B$  for  $BC$  (5)

$$W \sin \theta = 2W a \cos \theta$$

$$\tan \theta = 2 \quad (5)$$

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Considering the equilibrium of BC,

$$\leftarrow X = W \quad (5)$$

$$\uparrow Y = W \quad (5)$$

$$\therefore R_B = \sqrt{W^2 + W^2} \quad (5)$$

$$= \sqrt{2} W$$

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$R_B$  makes an angle of  $\tan^{-1} 1 = \frac{\pi}{4}$  with the horizontal  $(5)$

AB  $\curvearrowright A$

$$X \cdot 2\lambda \cos \theta = W\lambda \sin \theta + y \cdot 2\lambda \sin \theta \quad (10)$$

$$W \cdot 2 \cos \theta = W \sin \theta + W \cdot 2 \sin \theta$$

$$\frac{2}{3} = \tan \theta$$

$$\theta = \tan^{-1} \left( \frac{2}{3} \right) \quad (5)$$

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By considering the equilibrium of AB ,

$$\rightarrow X_1 = X = W \quad (5)$$

$$\uparrow Y_1 = 2W \quad (5)$$

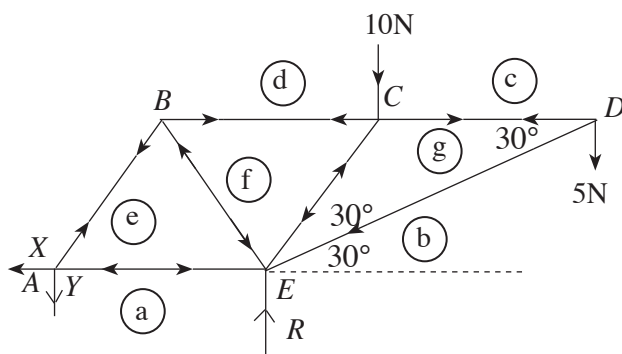
$$\therefore R_A = \sqrt{X_1^2 + Y_1^2}$$

$$= \sqrt{5} W \quad (5)$$

$R_A$  makes an angle of  $\tan^{-1}(2)$  with the horizontal  $(5)$

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(b)



Let the length of a rod (other than DE) be  $2a$ .

(i) Considering the system,  $\curvearrowright A$

$$R \cdot 2a - 10 \times 3a - 5 \times 5a = 0 \quad (5)$$

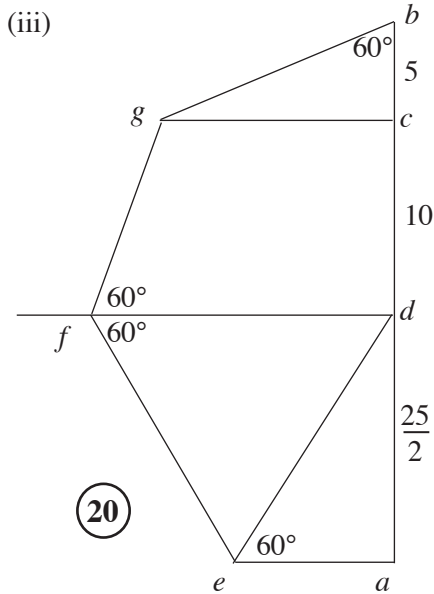
$$R = \frac{55}{2} \text{ N} \quad (5)$$

$$\therefore \text{the vertical force acting at } E = \frac{55}{2} \text{ N}$$

(ii)  $\uparrow -Y + R - 10 - 5 = 0$   
 $-Y = 15 - \frac{55}{2} = -\frac{25}{2} \text{ N}$   
 $\therefore Y = \frac{25}{2} \text{ N}$   
 $\leftarrow X = 0$

The vertical component of the reaction at the hinge A =  $\downarrow \frac{25}{2} \text{ N}$  (5)  
 Horizontal component = 0 (5)

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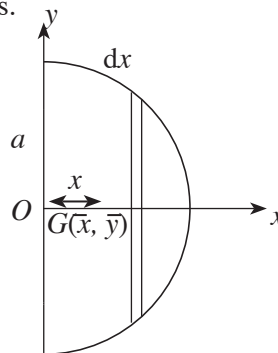
Rod	Magnitude	Stress
AE	$\frac{25\sqrt{3}}{6} \text{ N}$	Thrust
AB	$\frac{25\sqrt{3}}{3} \text{ N}$	Tension
BE	$\frac{25\sqrt{3}}{3} \text{ N}$	Thrust
BC	$\frac{25\sqrt{3}}{3} \text{ N}$	Tension
CE	$\frac{20\sqrt{3}}{3} \text{ N}$	Thrust
CD	$5\sqrt{3}$	Tension
ED	10 N	Thrust

35

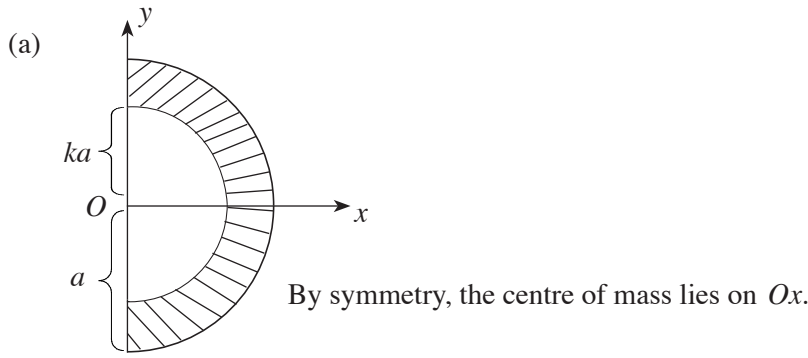
55

16. By symmetry, the centre of mass lies on the x axis.

$\bar{y} = 0$  (5)  
 $\bar{x} = \frac{\int_0^a \pi \rho x (a^2 - x^2) dx}{\int_0^a \pi \rho (a^2 - x^2) dx}$  (10)  
 $= \frac{\left[ \frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a}{\left[ a^2 x - \frac{x^3}{3} \right]_0^a}$  (5)  
 $= \frac{3a}{8}$  (5)  
 $\therefore G \equiv \left( \frac{3a}{8}, 0 \right)$



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Object	Mass	Distance from $O$ to the centre of mass
Hemisphere	$\frac{2}{3} \pi a^3 \rho$	$\frac{3a}{8}$
Hemisphere which is removed	$\frac{2}{3} \pi (ka)^3 \rho$	$\frac{3ka}{8}$
Remaining portion	$\frac{2}{3} \pi a^3 \rho (1 - k^3)$	$\bar{x}$

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5

$$\bar{x} = \frac{\frac{2}{3} \pi a^3 \rho \frac{3a}{8} - \frac{2}{3} \pi k^3 a^3 \rho \frac{3ka}{8}}{\frac{2}{3} \pi a^3 \rho (1 - k^3)} \quad (15)$$

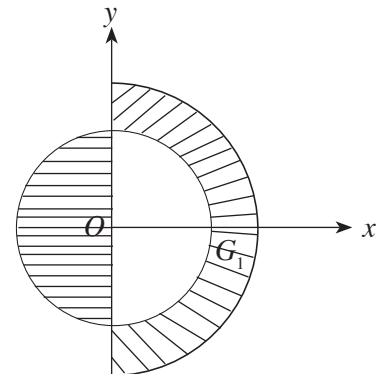
$$= \frac{\frac{3a}{8} (1 - k^4)}{(1 - k^3)} = \frac{3a (1 + k^2) (1 - k) (1 + k)}{8 (1 - k) (1 + k + k^2)}$$

$$= \frac{3a (1 + k^2) (1 + k)}{8 (1 + k + k^2)} \quad (10)$$

40

(b) Let the centre of mass be  $G_1(\bar{x}_1, \bar{y}_1)$

When the hemisphere that is removed is attached to the remaining portion as shown in the figure, due to symmetry about  $Ox$  we have  $\bar{y}_1 = 0$ . (5)



(i) Let the mass of the hemisphere that was removed be  $m$ , and the mass of the hemisphere of radius  $a$  be  $M$ .

$$\frac{m}{M} = \frac{\frac{2\pi}{3} k^3 a^3 \rho}{\frac{2\pi}{3} a^3 \rho} = k^3 \quad (5)$$

$$m = Mk^3 \quad (5)$$

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(ii) The distance from  $O$  to the centre of mass of the composite object is  $\bar{x}_1$ .

$$\bar{x}_1 = \frac{(M-m)\bar{x} + m\left(-\frac{3}{8}ka\right)}{(M-m) + m} \quad (15)$$

Since  $(M-m)\bar{x} = M\left(\frac{3a}{8}\right) - m\left(\frac{3ka}{8}\right)$ , (5)

$$\begin{aligned} \bar{x}_1 &= \frac{M\left(\frac{3a}{8}\right) - m\left(\frac{3ka}{8}\right) - m\left(\frac{3ka}{8}\right)}{M} \\ &= \frac{3a}{8} \frac{(M-2mk)}{M} \\ &= \frac{3a}{8} \left(1 - \frac{2m}{M}k\right) \end{aligned} \quad (10)$$

The distance from  $O$  to the centre of mass of the composite object  $= \frac{3a}{8} (1 - 2k^4)$

30

(iii) The centre of mass  $G_1$  should coincide (5)

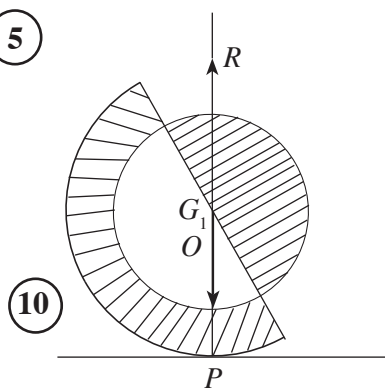
with  $O$ . That is,  $\bar{x}_1 = 0$ . (5)

$$\frac{3a}{8} (1 - 2k^4) = 0 \quad (5)$$

$$2k^4 = 1$$

$$k^2 = \pm \frac{1}{2} \quad (5)$$

Since  $k^2 > 0$ ,  $k^2 = \frac{1}{2}$



(10)

30

17. (a)  $P(A) = 0.1$ ,  $P(A \cup B) = 0.37$  and  $P(C) = 0.2$

(i)  $P(A \cup B) = P(A) + P(B) - P(A)P(B)$  (5)

(Since  $A$  and  $B$  are independent)

$$0.37 = 0.1 + P(B) - 0.1P(B)$$

$$0.37 - 0.1 = 0.9P(B)$$

$$0.3 = P(B)$$

15

(ii)  $P(B' | A') = \frac{P(B' \cap A')}{P(A')}$  (5)

Here  $P(B' \cap A') = P[(B \cup A)'] = 1 - P(A \cup B)$  (5)

$$= 1 - 0.37 = 0.63$$

$$P(A') = 1 - P(A) = 1 - 0.1 = 0.9$$

$$\therefore P(B' | A') = \frac{0.63}{0.9} = 0.7$$

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$$\begin{aligned} \text{(iii) } P(A' \cap B' \cap C) &= P(A') P(B') P(C) && \textcircled{5} \\ &= 0.9 \times 0.7 \times 0.2 \\ &= 0.126 && \textcircled{5} \end{aligned}$$

10

$$\text{(iv) } X : (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \quad \textcircled{5}$$

$$\begin{aligned} P(X) &= P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C) \\ &= P(A) P(B') P(C') + P(A') P(B) P(C') + P(A') P(B') P(C) && \textcircled{5} \\ &= 0.1 \times 0.7 \times 0.8 + 0.9 \times 0.3 \times 0.8 + 0.9 \times 0.7 \times 0.2 && \textcircled{10} \\ &= 0.398 \end{aligned}$$

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$$\begin{aligned} \Rightarrow P(A | X) &= \frac{P(A \cap X)}{P(X)} \\ &= \frac{P(A \cap B' \cap C')}{P(X)} && \textcircled{5} \\ &= \frac{0.1 \times 0.7 \times 0.8}{0.398} \\ &= \frac{56}{398} && \textcircled{5} \\ &= \frac{28}{199} \end{aligned}$$

10

$$\text{(b) (i) } (\alpha) \text{ Mean } 8 = \frac{\sum_{r=1}^n x_r}{n} \quad \textcircled{5}$$

$$\begin{aligned} \text{Mean value of the marks } \bar{x} &= \frac{28 + 56 + 23 + 94 + 8 + 5 + 13 + 846}{28} && \textcircled{5} \\ &= \frac{1073}{28} \\ &= 38.32 && \textcircled{5} \end{aligned}$$

15

(β) Since the number 94 should have been 49;

Thus the drop of mark = 45

Since the number 05 should have been 50;

Thus the rise of mark = 45

Therefore no change in the mean value.

5

5

10

$$\text{Standard deviation} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} \quad (5)$$

$$\begin{aligned} \text{Variance } s_x^2 &= \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \\ &= \frac{\sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n} \quad (5) \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - 2\bar{x} \frac{\sum x_i}{n} + \bar{x}^2 \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2 \quad (5) \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2 \end{aligned}$$

Let  $X = \{x_1, x_2, \dots, x_{20}\}$  and  $Y = \{y_1, y_2, \dots, y_{10}\}$ .

$$\text{Since } \sum_{i=1}^{20} x_i = 320, \quad \sum_{i=1}^{20} x_i^2 = 5840 \quad (5)$$

and

$$\sum_{i=1}^{10} y_i = 130, \quad \sum_{i=1}^{10} y_i^2 = 2380 \quad (5)$$

$$\therefore \bar{x} = \frac{\sum_{i=1}^{20} x_i}{20} = \frac{320}{20} = 16 \quad (5)$$

$$\text{and } s_x^2 = \frac{\sum_{i=1}^{20} x_i^2}{20} - 16^2 = \frac{5840}{20} - 16^2$$

$$= 292 - 256 = 36$$

$$\therefore s_x = 6 \quad (5)$$

$$\bar{y} = \frac{\sum_{i=1}^{10} y_i}{10} = \frac{130}{10} = 13 \quad (5)$$

$$s_y^2 = \frac{\sum_{i=1}^{10} y_i^2}{10} - 13^2 = \frac{2380}{10} - 169 = 69$$

$$\therefore s_y = 8.30 \quad (5)$$

Let  $Z = X \cup Y$ .

$$\begin{aligned} \bar{z} &= \frac{\sum_{i=1}^{20} x_i + \sum_{i=1}^{10} y_i}{30} \\ &= \frac{320 + 130}{30} = 15 \quad (5) \end{aligned}$$



$$s_z^2 = \frac{\sum_{i=1}^{20} x_i^2 + \sum_{i=1}^{10} y_i^2}{30} - \bar{z}^2 \quad (5)$$

$$= 274 - 225 = 49$$

$$s_z = 7 \quad (5)$$

60