01. $1+2+3+4+\cdots+(2n-1)+2n = n(2n+1)$

When $n=1$, $L.H.S = 1+2 = 3$

$R.H.S = 1\cdot 3 = 3$

$\therefore\ L.H.S = R.H.S$

The result is true for $n=1$. 5

Let the result is true for $n=p$.

$1+2+3+4+\cdots+(2p-1)+2p = p(2p+1)$ 5

when $n=p+1$,

$1+2+3+4+\cdots+2p+(2p+1)+(2p+2)$

$= p(2p+1)+(2p+1)+(2p+2)$ 5

$= 2p^2+5p+3$

$= (p+1)(2p+3) = (p+1)\big(2(p+1)+1\big)$ 5

$\therefore$ The result is true for $n=p+1$.

Therefore, by the Principle of Mathematical Induction, the result is true. 5

25

02. $x^2 - 5|x| - 6 < 0$

case (1) $x < 0$ 5

$x^2+5x-6<0$

$(x+6)(x-1)<0$

$-6<x<1$

$\therefore\ -6<x<0$ -------- (1) 5

case (2) $x \geqslant 0$ 5

$x^2-5x-6<0$

$(x-6)(x+1)<0$

$-1<x<6$

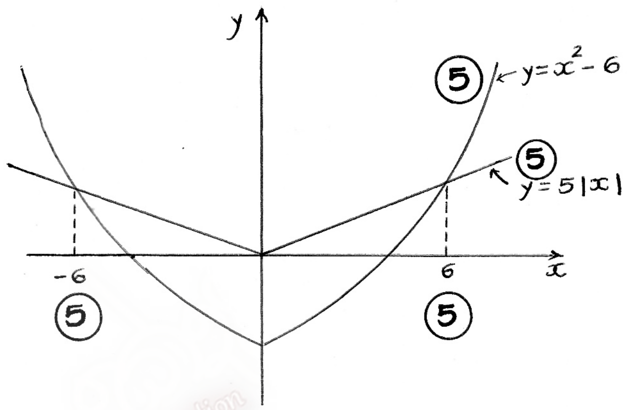
$\therefore\ 0 \leq x < 6$ ---- (2) 5

(1), (2) $\Rightarrow\ -6<x<6$ 5

25

Aliter

$x^2 - 6 < 5|x|$

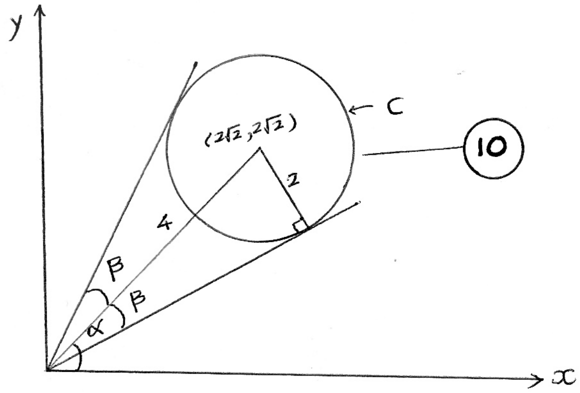



Solution is $-6<x<6$.

5

25

03. $|z-(2\sqrt{2}+2\sqrt{2}i)| = 2$




The circle with centre $(2\sqrt{2}, 2\sqrt{2})$ and radius 2.

$\sin\beta = \frac{1}{2} \Rightarrow \beta = \frac{\pi}{6}$

$\tan\alpha = \frac{2\sqrt{2}}{2\sqrt{2}} = 1 \Rightarrow \alpha = \frac{\pi}{4}$ 5

$Arg(z)|_{max} = \alpha+\beta = \frac{5\pi}{12}$ 5

$Arg(z)|_{min} = \alpha-\beta = \frac{\pi}{12}$ 5

25

04. ${}^nC_0(-3)^0 + {}^nC_1(-3)^1 + {}^nC_2(-3)^2 = 559$ (5)

$1 - 3n + \frac{9}{2}n(n-1) = 559$

$3n^2 - 5n - 372 = 0$

$(n-12)(3n+31) = 0$

$n = 12$ (5)

$T_{r+1} = {}^{12}C_r\, x^{12-r}\left(\frac{-3}{x^2}\right)^r$

$= {}^{12}C_r(-3)^r x^{12-3r}$ (5)

$12 - 3r = 0 \Rightarrow r = 4$ (5)

The term independent of $x$

$= {}^{12}C_4(-3)^4 = 40\,095$ (5)

25

05. $\lim\limits_{x\to\pi} \dfrac{\cos(\frac{\pi}{4}+\frac{x}{2}) + \cos(\frac{\pi}{4}-\frac{x}{2})}{x^2 - \pi^2}$

$= \lim\limits_{x\to\pi} \dfrac{2\cos\frac{\pi}{4}\cos\frac{x}{2}}{(x-\pi)(x+\pi)}$ (5)

$= \lim\limits_{x\to\pi} \dfrac{\sqrt{2}\sin(\frac{\pi}{2}-\frac{x}{2})}{-2(\frac{\pi}{2}-\frac{x}{2})(x+\pi)}$ (10)

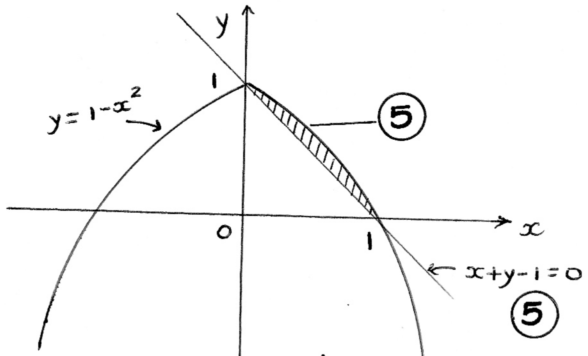
$= \lim\limits_{(\frac{\pi}{2}-\frac{x}{2})\to 0} \dfrac{\sin(\frac{\pi}{2}-\frac{x}{2})}{\frac{\pi}{2}-\frac{x}{2}} \cdot \lim\limits_{x\to\pi} \dfrac{\sqrt{2}}{-2(x+\pi)}$ (5)

$= 1 \times \dfrac{\sqrt{2}}{-2.(2\pi)}$ (5)

$= -\dfrac{1}{2\sqrt{2}\pi}$

25

06.




Area $= \int_0^1 (1-x^2)\,dx - \int_0^1 (1-x)\,dx$ (5)

$= \int_0^1 (x - x^2)\,dx$

$= \left\{\frac{x^2}{2} - \frac{x^3}{3}\right\}_0^1$ (5)

$= \frac{1}{2} - \frac{1}{3}$ (5)

$= \frac{1}{6}$

25

07. $y = \dfrac{2}{1+x^2}$

$\dfrac{dy}{dx} = 2(-1)(1+x^2)^{-2}2x = \dfrac{-4x}{(1+x^2)^2}$ (5)

$\left(\dfrac{dy}{dx}\right)_{(1,1)} = -2$ (5)

Equation of the tangent

$y - 1 = -2(x-1)$ (5)

$2x + y - 3 = 0$

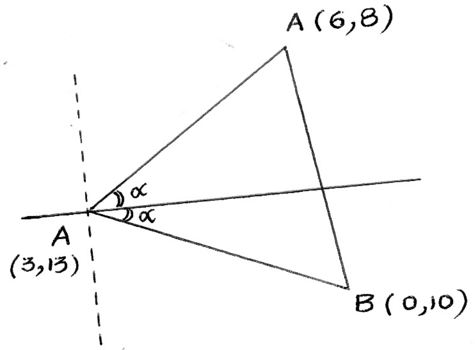
$\left(\dfrac{dy}{dx}\right)_{(0,2)} = 0$

Equation of the normal $y = 2$ (5)

* Point of intersection $(\frac{1}{2}, 2)$ (5)

25

08.




Equation of AB, $y - x = 10$ (5)

Equation of AC, $y + 7x = 50$ (5)

Equation of bisector of $B\hat{A}C$,

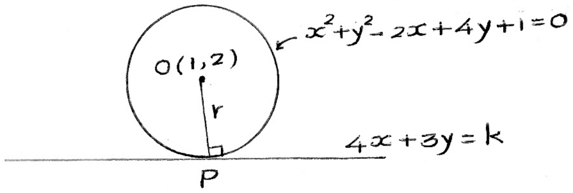
$\dfrac{y-x-10}{\sqrt{2}} = \pm\dfrac{y+7x-50}{5\sqrt{2}}$ (10)

$(+) \Rightarrow\ y - 3x = 0$

$(-) \Rightarrow\ 3y + x = 50$ (5)

25

09.




The tangent can be written in the form $4x+3y=k$. 5

OP = r

$$\frac{|4\times1+3\times(-2)-k|}{\sqrt{4^2+3^2}} = \sqrt{1^2+(-2)^2-1}$$ 5 5

$$\Rightarrow \frac{|-2-k|}{5} = 2 \Rightarrow k = -12, 8$$ 5 5

$\therefore$ Equations of the tangents

$4x+3y=8$ , $4x+3y+12=0$

25

10. $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\Rightarrow b^2-2bc\cos A+c^2-a^2=0$ 5

$b_1+b_2 = 2c\cos A$ , $b_1 b_2 = c^2-a^2$ 5

Put $b_2 = 2b_1$

$3b_1 = 2c\cos A$ & $2b_1^2 = c^2-a^2$ 5

$\Rightarrow 2\left(\dfrac{2c\cos A}{3}\right)^2 = c^2-a^2$ 5

$\Rightarrow 8c^2(1-\sin^2 A) = 9c^2-9a^2$ 5

$\Rightarrow \sin A = \sqrt{\dfrac{9a^2-c^2}{8c^2}}$

25

11. (a) $f(x) = (\lambda-x)^3+(\mu-x)^3-(\lambda+\mu-2x)^3$

$f(\lambda) = (\lambda-\lambda)^3+(\mu-\lambda)^3-(\lambda+\mu-2\lambda)^3$

$= 0+(\mu-\lambda)^3-(\mu-\lambda)^3$ 10

$= 0$

$\therefore$ $x-\lambda$ is a factor of $f(x)$. 5

$f(\mu) = (\lambda-\mu)^3+(\mu-\mu)^3-(\lambda+\mu-2\mu)^3$

$= (\lambda-\mu)^3-(\lambda-\mu)^3$ 10

$= 0$

$\therefore$ $x-\mu$ is a factor of $f(x)$. 5

$f(x) = (x-\lambda)(x-\mu)(Ax+B)$ 5

$(\lambda-x)^3+(\mu-x)^3-(\lambda+\mu-2x)^3 \equiv (x-\lambda)(x-\mu)(Ax+B)$

$\lambda^3-3\lambda^2x+3\lambda x^2-x^3+\mu^3-3\mu^2x+3\mu x^2-x^3$
$-\left((\lambda+\mu)^3-3(\lambda+\mu)^2 2x+3(\lambda+\mu)(2x)^2-(2x)^3\right)$
$\equiv (x-\lambda)(x-\mu)(Ax+B)$

$x^3$ // $-1-1+8=A$ 5

$A=6$ 5

$x^0$ // $\lambda^3+\mu^3-(\lambda+\mu)^3 = B\lambda\mu$ 5

$\lambda^3+\mu^3-(\lambda^3+3\lambda^2\mu+3\lambda\mu^2+\mu^3) = B\lambda\mu$

$-3\lambda\mu(\lambda+\mu) = B\lambda\mu$

$B = -3(\lambda+\mu)$ 5

$f(x) \equiv (x-\lambda)(x-\mu)(6x-3(\lambda+\mu))$

$= 3(x-\lambda)(x-\mu)(2x-\lambda-\mu)$ 5

$f(x) = (x-\lambda-\mu)\phi(x)+R$

$f(\lambda+\mu) = R$ 5

Remainder $= (\lambda-\lambda-\mu)^3+(\mu-\lambda-\mu)^3-(\lambda+\mu-2\lambda-2\mu)^3$

$= -\mu^3-\lambda^3+(\lambda+\mu)^3$ 5

$= -\mu^3-\lambda^3+\lambda^3+3\lambda^2\mu+3\lambda\mu^2+\mu^3$

$= 3\lambda\mu(\lambda+\mu)$ 5

75

(b) $g(x) = ax^2+bx+c$

$= a\left\{\left(x+\dfrac{b}{2a}\right)^2-\left(\dfrac{b^2-4ac}{4a^2}\right)\right\}$ 5

$= a\left(x+\dfrac{b}{2a}\right)^2-\left(\dfrac{b^2-4ac}{4a}\right)$ 5

If $a<0$ , $b^2-4ac>0$

$g(x)\big|_{max} = -\left(\dfrac{b^2-4ac}{4a}\right) > 0$ 5

$\therefore$ $g(x)$ has positive maximum value.

Also, $g(x)\Big|_{max} = \dfrac{4ac-b^2}{4a}$ 5

$f(x) = x^2 + kx^2 - kx + 2k$

$= (1+k)x^2 - kx + 2k$ 5

If $f(x)$ has positive maximum value ; $1+k<0$ , $k^2 - 4(1+k)2k > 0$ 5

$k<-1$ , $k(7k+8)<0$

$k<-1$ , $-\frac{8}{7} < k < 0$ 5

$\therefore$ $-\frac{8}{7} < k < -1$ 5

The roots of $f(x) = (1+k)x^2 - kx + 2k = 0$

are $\alpha$ , $\beta$.

$\alpha+\beta = \dfrac{k}{1+k}$ , $\alpha\beta = \dfrac{2k}{1+k}$ 10

$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \dfrac{-k(3k+4)}{(1+k)^2}$ 10

$\alpha^2\beta^2 = \dfrac{4k^2}{(1+k)^2}$ 5

Function is $\lambda\left(x^2 + \dfrac{k(3k+4)}{(1+k)^2}x + \dfrac{4k^2}{(1+k)^2}\right)$ 5

Since this passes through $(0, \alpha^2\beta^2)$ , $\lambda = 1$

Function is $x^2 + \dfrac{k(3k+4)}{(1+k)^2}x + \dfrac{4k^2}{(1+k)^2}$ 5

75

12. 

$$\begin{array}{cccc} N & I & E & T \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 3 & 1 \end{array}$$

Number of ways which can be made having all letters is

$\dfrac{8!}{3!\,3!} = 1120$ 10

$$\overset{N}{\downarrow}\ \_\ \boxed{EEE}\ \overset{N}{\downarrow}\ \_\ I\ \_\ T\ \overset{N}{\downarrow}\ \_$$

${}^4C_3 \cdot 3!$ 10

$= 24$ 5

| case | | | No. of combinations |
|---|---|---|---|
| (1) 3 same | 1 different | ${}^2C_1\,{}^3C_1\,\frac{4!}{3!} = 24$ 5 | |
| (2) 2 same | 2 another same | ${}^2C_2\,\frac{4!}{2!\,2!} = 6$ 5 | |
| (3) 2 same | 2 different | ${}^2C_1\,{}^3C_2\,\frac{4!}{2!} = 72$ 5 | |
| (4) all | different | ${}^4C_4\ 4! = 24$ 5 | |

Total $= 24+6+72+24 = 126$ 10

$$\begin{array}{cccc} N & I & E & T \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 1 & 1 \end{array}$$

In the letters, number of permitations of 2 letters one time :

case

(1) 2 same ${}^1C_1\ \frac{2!}{2!} = 1$ 5

(2) different ${}^4C_2\ 2! = 12$ 5

$\therefore$ No. of arrangements beginning from E and ending of E is

$1 + 12 = 13$ 5

70

(b) $x^2 + 2x + 2 \equiv Ax(x+1) + B(x+1) + Cx$

$x = 0 \Rightarrow 2 = B$ 5

$x = -1 \Rightarrow 1 = -C \Rightarrow C = -1$ 5

$x^2 \,//$ $1 = A$ 5

$x^2 + 2x + 2 \equiv x(x+1) + 2(x+1) - x$

$U_r = \dfrac{(r+1)^2 + 1}{r(r+1)}\left(\frac{1}{2}\right)^r$ 5

$$u_r = \left(\frac{r(r+1)+2(r+1)-r}{r(r+1)}\right)\left(\frac{1}{2}\right)^r \quad 5$$

$$= \left(1+\frac{2}{r}-\frac{1}{r+1}\right)\left(\frac{1}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^r + \frac{1}{r}\left(\frac{1}{2}\right)^{r-1} - \frac{1}{r+1}\left(\frac{1}{2}\right)^r \quad 5$$

$$= \left(\frac{1}{2}\right)^r + f(r) - f(r+1) \quad 5$$

where $f(r) = \frac{1}{r}\left(\frac{1}{2}\right)^{r-1}$ 5

$r=1 \quad u_1 = \left(\frac{1}{2}\right)^1 + f(1) - \cancel{f(2)}$ 5

$r=2 \quad u_2 = \left(\frac{1}{2}\right)^2 + \cancel{f(2)} - \cancel{f(3)}$

$r=n-1 \quad u_{n-1} = \left(\frac{1}{2}\right)^{n-1} + \cancel{f(n-1)} - \cancel{f(n)}$ 5

$r=n \quad u_n = \left(\frac{1}{2}\right)^n + \cancel{f(n)} - f(n+1)$

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left(\frac{1}{2}\right)^r + f(1) - f(n+1) \quad 5$$

$$= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} + 1 - \frac{1}{n+1}\left(\frac{1}{2}\right)^n \quad 10$$

$$= 2 - \left(\frac{1}{2}\right)^n - \frac{1}{n+1}\left(\frac{1}{2}\right)^n$$

$$= 2 - \left(\frac{1}{2}\right)^n\left(1+\frac{1}{n+1}\right)$$

$$= 2 - \left(\frac{1}{2}\right)^n\left(\frac{n+2}{n+1}\right) \quad 5$$

$$\lim_{n\to\infty}\sum_{r=1}^{n} u_r = \lim_{n\to\infty}\left\{2-\left(\frac{1}{2}\right)^n\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)\right\} = 2 \quad 5$$

$\therefore$ The series is convergent

$$\sum_{r=1}^{\infty} u_r = 2 \quad 5$$

80

13.(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left.\begin{array}{l} ap+br=1 \;\cdots\cdots(1) \\ aq+bs=0 \;\cdots\cdots(2) \\ cp+dr=0 \;\cdots\cdots(3) \\ cq+ds=1 \;\cdots\cdots(4) \end{array}\right\} \quad 10$$

(1), (3) $\Rightarrow$ $a\left(-\frac{dr}{c}\right)+br=1$

$$r = \frac{c}{bc-ad} = \frac{-c}{ad-bc} \quad 5$$

$$p = \frac{-d}{bc-ad} = \frac{d}{ad-bc} \quad 5$$

(2), (4) $\Rightarrow$ $c\left(-\frac{bs}{a}\right)+ds=1$

$$s = \frac{a}{ad-bc} \quad 5$$

$$q = \frac{-bs}{a} = -\frac{b}{a}\,\frac{a}{ad-bc} = \frac{-b}{ad-bc} \quad 5$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

$$= \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad 5$$

(i) $A^{-1} = \frac{1}{2+15}\begin{pmatrix} 1 & -5 \\ 3 & 2 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 1 & -5 \\ 3 & 2 \end{pmatrix}$ 5

$B^{-1} = \frac{1}{6+2}\begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$ 5

(ii) $AXB = I$

$A^{-1}AXB = A^{-1}I$

$IXB = A^{-1}$

$XB = A^{-1}$

$XBB^{-1} = A^{-1}B^{-1}$

$XI = A^{-1}B^{-1}$

$X = A^{-1}B^{-1}$ 5

$$= \frac{1}{17}\begin{pmatrix} 1 & -5 \\ 3 & 2 \end{pmatrix} \cdot \frac{1}{8}\begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{136}\begin{pmatrix} 1 & -5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{136}\begin{pmatrix} 8 & -8 \\ 7 & 10 \end{pmatrix} \qquad 5$$

(iii) $2x + 5y = 12$

$-3x + y = -1$

$$\begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \end{pmatrix} \qquad 10$$

$A \; Y = C$, where

$$Y = \begin{pmatrix} x \\ y \end{pmatrix}, \quad C = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$$

$$Y = A^{-1}C = \frac{1}{17}\begin{pmatrix} 1 & -5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \quad 5$$

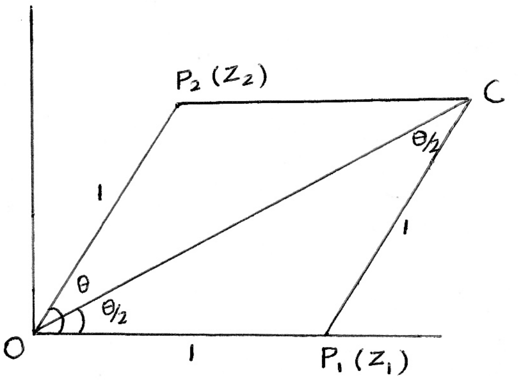
$$= \frac{1}{17}\begin{pmatrix} 17 \\ 34 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$x = 1$

$y = 2$ 5

75

(b) Finding the point C 10




5

In ▱ $OP_1CP_2$, $OP_1 = OP_2 = 1$

$\therefore$ $OP_1CP_2$ is a rhombus. 5

$P_1\hat{O}C = C\hat{O}P_2 = \frac{\theta}{2}$ 5

$OC = 2\cos\frac{\theta}{2}$

$|z_1+z_2| = 2\cos\frac{\theta}{2}$ 5

$\text{Arg}(z_1+z_2) = \frac{\theta}{2}$ 5

$|z_1+z_2|_{max} = 2$ when $\theta = 0$ 5

$z_2 = \cos 0 + i\sin 0 = 1$ 5

$|z_1+z_2|_{min} = 0$ when $\theta = \pi$ 5

$z_2 = \cos\pi + i\sin\pi = -1$ 5

$$\frac{1}{z_1+z_2} = \frac{1}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} \quad 5$$

$$= \frac{1}{2}\sec\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) \quad 5$$

$$= \frac{1}{2}\sec\frac{\theta}{2}\left(\cos\left(-\frac{\theta}{2}\right) + i\sin\left(-\frac{\theta}{2}\right)\right) \quad 5$$

$$\text{Re}\left(\frac{1}{z_1+z_2}\right) = \frac{1}{2}\sec\frac{\theta}{2} \quad 5$$

75

14. (a) $y = a\cos(\ln x) + b\sin(\ln x)$

$$\frac{dy}{dx} = -a\sin(\ln x)\cdot\frac{1}{x} + b\cos(\ln x)\cdot\frac{1}{x} \quad 10$$

$$x\frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

$$x\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx} = -\left\{\frac{a\cos(\ln x)}{x} + \frac{b\sin(\ln x)}{x}\right\} \quad 10$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \quad 5$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

25

(b) $$f(x) = \frac{2x^2}{(x+2)(x-4)} = \frac{2x^2}{x^2-2x-8}$$

$$f'(x) = \frac{(x^2-2x-8)4x - 2x^2(2x-2)}{(x^2-2x-8)^2} \quad 10$$

$$= \frac{2x\{2x^2-4x-16-2x^2+2x\}}{(x^2-2x-8)^2}$$

$$= \frac{2x\{-2x-16\}}{(x^2-2x-8)^2} \quad 5$$

$$= \frac{-4x(x+8)}{(x+2)^2(x-4)^2}$$

At the turning point $f'(x)=0$ 5

$\Rightarrow x=0$ or $x=-8$ 5

$x=-2$ , $x=4$ are vertical asymptotes

| Range of $x$ | $x<-8$ | $-8<x<-2$ | $-2<x<0$ | $0<x<4$ | $x>4$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$ | $(-)$ | $(+)$ | $(+)$ | $(-)$ | $(-)$ |

15

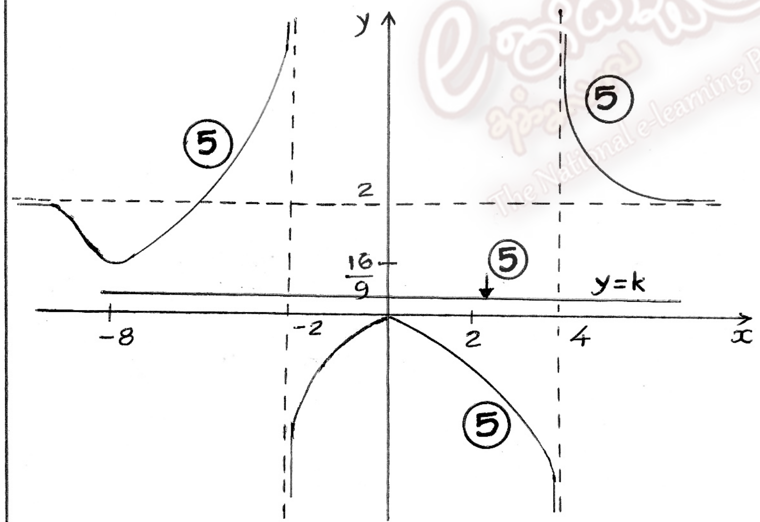
At $x=-8$ min ; minimum point $\equiv \left(-8, \frac{16}{9}\right)$ 5

At $x=0$ max ; maximum point $\equiv (0,0)$ 5

$$y=\frac{2x^2}{(x+2)(x-4)} = \frac{2}{\left(1+\frac{2}{x}\right)\left(1-\frac{4}{x}\right)}$$

$x \to \pm\infty \quad y \to 2$ 5

$x=0 \Rightarrow y=0 \qquad (0,0)$




$$2x^2 - k(x+2)(x-4)=0$$

$$\frac{2x^2}{(x+2)(x-4)} = k \quad 5$$

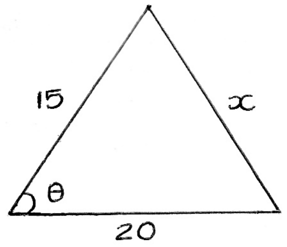
The solutions of the equation are $x$- coordinates of the points of intersection of the given curve and the straight line $y=k$.

If $0<k<\frac{16}{9}$ , both curves will not intersect.

The equation has no real solution. 5

90

(c)




$$\frac{d\theta}{dt} = \frac{\pi}{90} \text{ rad/s}$$

By cos rule ,

$$x^2 = 20^2 + 15^2 - 2\cdot 15\cdot 20 \cos\theta \quad 10$$

$$\theta=\frac{\pi}{3} \Rightarrow x^2 = 325$$

$$x = 5\sqrt{13} \quad 5$$

$$2x\frac{dx}{dt} = -600(-\sin\theta)\frac{d\theta}{dt} \quad 10$$

$$\frac{dx}{dt} = \frac{300\sin\theta}{x}\cdot\frac{d\theta}{dt} \quad 5$$

$$\left(\frac{dx}{dt}\right)_{\theta=\frac{\pi}{3}} = \frac{300\times\frac{\sqrt{3}}{2}}{5\sqrt{13}}\cdot\frac{\pi}{90} \quad 5$$

$$= \frac{\pi}{\sqrt{39}} \text{ units/s}$$

35

15. (a) $\int_0^a f(x)\,dx$

Let $y=a-x$

$$\left.\begin{aligned} \frac{dy}{dx} &= -1 \\ x=0 &\Rightarrow y=a \\ x=a &\Rightarrow y=0 \end{aligned}\right\} \quad 5$$

$$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(y)(-dy) \quad 5$$

$$= \int_0^a f(y)\,dy \quad 5$$

$$= \int_0^a f(x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \int_0^{\pi} (\pi - x) f(\sin(\pi - x))\,dx \quad 5$$

$$= \int_0^{\pi} (\pi - x) f(\sin x)\,dx \quad 5$$

$$= \pi\int_0^{\pi} f(\sin x)\,dx - \int_0^{\pi} x f(\sin x)\,dx$$

$$\Rightarrow\ 2\int_0^{\pi} x f(\sin x)\,dx = \pi\int_0^{\pi} f(\sin x)\,dx \quad 5$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{x \sin x}{2-\sin^2 x}\,dx \quad 5$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{2-\sin^2 x}\,dx \quad 5 \quad (\because \text{ by part (a)})$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\,d(\cos x) \quad 5$$

$$= -\frac{\pi}{2}\left\{\tan^{-1}(\cos x)\right\}_0^{\pi} \quad 5$$

$$= -\frac{\pi}{2}\left\{\tan^{-1}(-1) - \tan^{-1} 1\right\} \quad 5$$

$$= \pi \tan^{-1}(1) \quad 5$$

$$= \frac{\pi^2}{4}$$

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(b) $\int x \ln(1+x^2)\,dx$

$$= \ln(1+x^2)\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{(1+x^2)}2x\,dx \quad 15$$

$$= \frac{x^2}{2}\ln(1+x^2) - \int \frac{x^3}{x^2+1}\,dx \quad 5$$

$$= \frac{x^2}{2}\ln(1+x^2) - \int \frac{x(x^2+1)-x}{x^2+1}\,dx \quad 5$$

$$= \frac{x^2}{2}\ln(1+x^2) - \int\left(x - \frac{x}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\ln(1+x^2) - \frac{x^2}{2} + \frac{1}{2}\ln(x^2+1) + C$$

5 5 5

C - constant

40

(c) $x^3 = \lambda(x-1)(4x^2+4x+2) + \mu(8x+4) + \gamma$

$x=1$ ; $1 = 12\mu + \gamma$

$x^3$ ; $1 = 4\lambda$

$x^0$ ; $0 = -2\lambda + 4\mu + \gamma$

$$\Rightarrow \lambda = \frac{1}{4}, \mu = \frac{1}{16}, \gamma = \frac{1}{4} \quad 15$$

$$\int \frac{x^3}{4x^2+4x+2}\,dx$$

$$= \frac{1}{4}\int (x-1)\,dx + \frac{1}{16}\int \frac{8x+4}{4x^2+4x+2}\,dx + \frac{1}{4\times 4}\int \frac{1}{x^2+x+\frac{1}{2}}\,dx \quad 5$$

$$= \frac{1}{4}\int (x-1)\,dx + \frac{1}{16}\int \frac{8x+4}{4x^2+4x+2}\,dx + \frac{1}{16}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{1}{2})^2}\,dx \quad 5$$

$$= \frac{1}{4}\left(\frac{x^2}{2} - x\right) + \frac{1}{16}\ln(4x^2+4x+2) + \frac{1}{8}\tan^{-1}(2x+1) + C$$

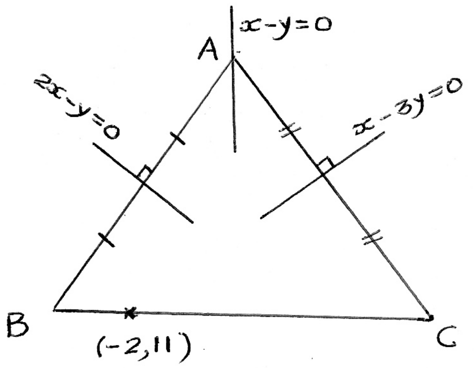
10 5 5 5

C - constant

50

16. (a) The coordinates of the mirror image are

$$\left(\alpha - \frac{2a(a\alpha+b\beta+c)}{a^2+b^2},\ \beta - \frac{2b(a\alpha+b\beta+c)}{a^2+b^2}\right)$$ 10




A lies on $x-y=0$.

$\therefore A \equiv (t, t)$ 5

The mirror image of A on $2x-y=0$ is B.

$$B \equiv \left(t - \frac{4(2t-t)}{4+1},\ t + \frac{2(2t-t)}{4+1}\right)$$ 5

$$B \equiv \left(\frac{t}{5}, \frac{7t}{5}\right)$$ 5

The mirror image of A on $y=0$ is C.

$$C \equiv \left(t - \frac{2(t-3t)}{1+9},\ t + \frac{6(t-3t)}{1+9}\right)$$ 5

$$C \equiv \left(\frac{7t}{5}, -\frac{t}{5}\right)$$ 5

$\therefore$ Equation of BC is

$$y - \frac{7t}{5} = \left(\frac{\frac{7t}{5}+\frac{t}{5}}{\frac{t}{5}-\frac{7t}{5}}\right)\left(x - \frac{t}{5}\right)$$ 5

$$4x + 3y - 5t = 0$$ 5

Since BC passes through $(-2, 11)$,

$$-8 + 33 - 5t = 0$$

$$t = 5$$ 5

$\therefore A \equiv (5,5)$ 5, $B \equiv (1,7)$ 5, $C \equiv (7,-1)$ 5

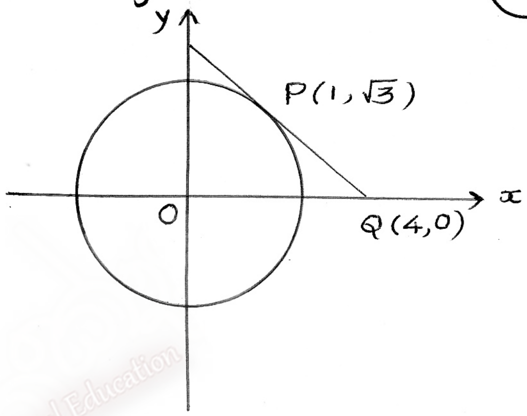
$AB \rightarrow x + 2y - 15 = 0$ 5

$BC \rightarrow 4x + 3y - 25 = 0$ 5

$AC \rightarrow 3x + y - 20 = 0$ 5

| 80 |
|---|

(b) Finding the equation of the tangent 20




Let $S \equiv x^2 + y^2 - 4 = 0$.

Equation of the tangent at the point $P(1, \sqrt{3})$ is

$$1 \cdot x + \sqrt{3} \cdot y = 4$$ 10

$$x + \sqrt{3}y = 4$$ 10

$$Q \equiv (4, 0)$$ 5

The normal at the point $P(1, \sqrt{3})$ is $\frac{x}{1} = \frac{y}{\sqrt{3}}$ 10

$$\sqrt{3}x - y = 0$$ 10

$\therefore$ The area of $\Delta$ OPQ is

$$\frac{1}{2} \times 4 \times \sqrt{3}$$

$$= 2\sqrt{3} \text{ square units}$$ 5

| 70 |
|---|

17. (a) $\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$

$(\cos x + \cos 3x) + \cos 2x = (\sin x + \sin 3x) + \sin 2x$ 5

$2\cos 2x \cos x + \cos 2x = 2\sin 2x \cos x + \sin 2x$ 10

$(2\cos x + 1)(\cos 2x - \sin 2x) = 0$ 5

$\cos x = -\frac{1}{2}$ or $\tan 2x = 1$ 5

| | |
|---|---|
| $x = 2n\pi \pm \frac{2\pi}{3}$ | $x = \frac{n\pi}{2} + \frac{\pi}{8}$ 5 |
| $n=0$ ; $x = -\frac{2\pi}{3}, \frac{2\pi}{3}$ | $n = 0, 1, 2, 3$ ; |
| ie. $\frac{4\pi}{3}, \frac{2\pi}{3}$ 5 | $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ 10 |

$\therefore$ The solutions between 0 and $2\pi$ are

$\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. 5

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(b) If $t = \tan\frac{\theta}{2}$

showing that $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ 10

$$\frac{1+\sin\theta}{3+2\cos\theta} = \frac{1+\frac{2t}{1+t^2}}{3+2\frac{(1-t^2)}{1+t^2}} = \frac{(1+t)^2}{5+t^2}$$ 5

$$E = \frac{(1+t)^2}{5+t^2}$$

$\Rightarrow (E-1)t^2 - 2t + 5E - 1 = 0$ 5

For all real values of $t$

$\Delta \geq 0$ 5

$(-2)^2 - 4(E-1)(5E-1) \geq 0$ 5

$E(5E-6) \leq 0$ 5

$0 \leq E \leq \frac{6}{5}$ 5

$$\therefore \quad 0 \leq \frac{1+\sin\theta}{3+2\cos\theta} \leq \frac{6}{5}$$

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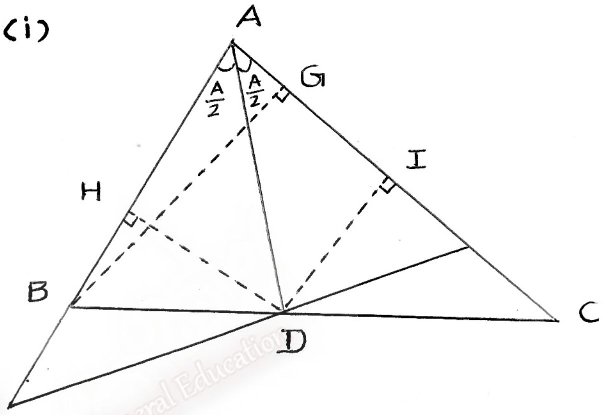
(c) (i)




Draw $BG \perp AC$

$\Delta ABC = \frac{1}{2} bc \sin A$ 10

Draw $DH \perp AB$

$\Delta ABD = \frac{1}{2} c \left(AD \sin\frac{A}{2}\right)$ 10

Draw $DI \perp AC$

$\Delta ACD = \frac{1}{2} b \left(AD \sin\frac{A}{2}\right)$ 10

But $\Delta ABC = \Delta ABD + \Delta ACD$ 5

$\frac{1}{2} bc \sin A = \frac{1}{2} c \left(AD \sin\frac{A}{2}\right) + \frac{1}{2} b \left(AD \sin\frac{A}{2}\right)$

$\Rightarrow AD = \frac{bc}{b+c} \frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{b+c} \cos\frac{A}{2}$ 5

(ii) In $\Delta ADE$, $AD = AE \cos\frac{A}{2}$ 5

$AE = AD \sec\frac{A}{2}$ 5

$= \frac{2bc}{b+c}$ 5

$\frac{2}{AE} = \frac{b+c}{bc}$ 5 $\Rightarrow \frac{2}{AE} = \frac{1}{b} + \frac{1}{c}$

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