G.C.E. (A. L.) Support Seminar - 2014

Combined Mathematics I

Three hours

Part A

Answer **all** the questions in the given space.

1. By using the Principle of Mathematical Induction, prove that $1+2+3+\cdots+n<\frac{1}{8}(2n+1)^2$ for all positive integers n.

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2. Find the solution set of the inequality $\frac{(x-a)(x-b)}{(x-c)} \le 0$ where a, b and c are real constants such that $a<b<c$.

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3. Find the number of arrangements that can be made by taking all the letters of the word FRACTION.

In how many of these arrangements are the vowels in even positions?

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4. Find the constants a and b such that $\lim_{x \to \infty}\left\{\frac{x^2+1}{x+1} - ax - b\right\} = 0$.

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5. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{x}\right) = a^{x}\ln a$; where $a \in \mathbb{R}^{+}$.

Hence, find $\int \frac{a^{x}}{1+a^{x}}\,\mathrm{d}x$ for $a \neq 1$.

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6. In the parallelogram *ABCD*, the equation of the side *AB* is $x + y + 2 = 0$ and the diagonals intersect at the point (4, 2). The straight line $x + 2 = 0$ passes through the point *A*. Find the equation of the side *DC*.

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7. A curve is given by the parametric equations $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$; where t is a parameter. Find the value of t corresponding to the point $(0, 1)$ and show that the tangent drawn to the curve at the said point is parallel to the x - axis.

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8. Find the equation of the circle of radius $2\sqrt{2}$ units which touches both the straight lines $x - y = 0$ and $7x - y = 0$, and lies entirely in the first quadrant.

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9. Find the complex number z such that $\text{Arg}\, iz = \pi$ and $|z+1|+|z-1|=4$.

10. Show that, if $\theta+\alpha=\frac{\pi}{6}$ then $(\sqrt{3}+\tan\theta)(\sqrt{3}+\tan\alpha)=4$.

Hence, deduce the value of $\tan\frac{\pi}{12}$.

Part B

Answer **five** questions only.

11. (i) If α and β are the roots of the quadratic equation $px^2+qx+r=0$, where p, q and r are real constants and $p \neq 0$, show that $\alpha+\beta=-\frac{q}{p}$ and $\alpha\beta=\frac{r}{p}$.

If the roots of the equation $x^2+ax+b=0$ are λ and 3μ, and the roots of the equation $x^2+cx+d=0$ are 3λ and μ, show that $b=d$. Here a, b, c and d are real constants.

Hence, show that the quadratic equation with roots λ and μ is $12x^2+3(a+c)x+4b=0$.

(ii) Let $f(x) \equiv x^3-2ax^2+\left(ab+a^2-b^2\right)x-ab(a-b)$. Here a and b are real constants such that $a \neq b$. Show that $x-a+b$ is a factor of the polynomial $f(x)$, and hence, solve the equation $f(x)=0$.

Deduce the values of the constants p, q and r in the equation $x^3+px^2+qx+r=0$, if 1, 3 and 4 are its roots.

(iii) Determine the partial fractions : $\dfrac{7x-10}{x^2(x-2)}$

12. (i) For any positive integral index n, write down the expansion of $(a+b)^n$. Here $a, b \in \mathbb{R}$.

Hence, deduce that $\sum_{r=0}^{n} {}^nC_r = 2^n$ and that $\sum_{r=0}^{n} \left({}^nC_r\right)^2 = {}^{2n}C_n$.

Show also that $\sum_{r=1}^{n} r\ {}^nC_r\, a^r\, b^{n-r} = na$ when $a+b=1$.

(ii) It is given that the sum of the first n terms of a series with r^{th} term U_r is $\sum_{r=1}^{n} U_r = \frac{n}{12}(n+1)(n+2)(n+3)$.
Here $1 \leq r \leq n$.

Find the function $f(r)$ and the constant k such that $\frac{1}{U_r} \equiv k\{f(r)-f(r+1)\}$, and thereby obtain $\sum_{r=1}^{n} \frac{1}{U_r}$.

Show that the infinite series $\sum_{r=1}^{\infty} \frac{1}{U_r}$ is convergent.

Deduce that $2 \leq 3\left\{1-\frac{2}{(n+1)(n+2)}\right\} < 3$.

13. (i) If $\mathbf{A}=\begin{pmatrix} 7 & 8 \\ -6 & -7 \end{pmatrix}$, find $\mathbf{A}^2$ and hence obtain $\mathbf{A}^{-1}$.

Determine the matrix $\mathbf{X}$ satisfying $\mathbf{A}^{2015}\mathbf{X}=\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$.

(ii) Obtain the roots of the equation $z^6=1$ in the form $x+iy$; where $x, y \in \mathbb{R}$ and $i^2=-1$. If z_1 and z_2 are two distinct roots of the equation $z^6=1$, using an Argand diagram, show that the values that $|z_1-z_2|$ can take are 1, 2 and $\sqrt{3}$.

(iii) If z is any complex number such that $|z|=\sqrt{3}$, using an Argand diagram, show that $2-\sqrt{3} \leq |z+2| \leq 2+\sqrt{3}$ and that $\frac{-\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{3}$.

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14. (i) If $x = \sec\theta + \tan\theta$ and $y = \operatorname{cosec}\theta + \cot\theta$, where θ is a parameter, show that $x + \frac{1}{x} = 2\sec\theta$ and $y + \frac{1}{y} = 2\operatorname{cosec}\theta$.

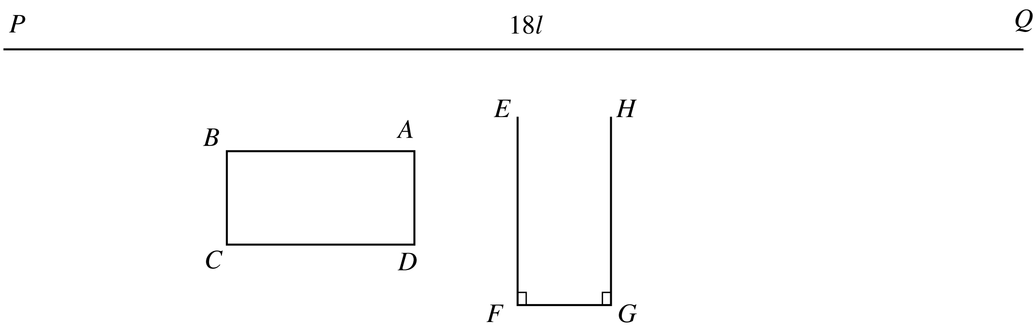
Find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

Hence, show that $\frac{dy}{dx} = -\frac{1+y^2}{1+x^2}$.

(ii) Using the principles of derivatives, show that $x - \frac{x^2}{2} + \frac{x^3}{3} > \ln(1+x)$ for $x > 0$.

Sketch the graph of the function $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ln(1+x)$ for $-1 < x \leq 1$, clearly indicating the asymptotes and stationary points. (Take that $\ln 2 \approx 0.69$)

(iii)



A student has been provided with a thin piece of wire PQ of length $18l$ to be used to prepare a structure of a pandol. The structure of the pandol is to be made by fixing the two unattached ends E and H of the frame $EFGH$ on to the side CD of the rectangular frame $ABCD$. Find separately, the lengths of the two pieces of wire that PQ should be cut into, so that $AB = EF = GH$, $AD = FG$ and the area of the structure of the pandol is maximized.

15. (i) By using a suitable substitution show that,

$$\int \sin^{-1}\left(\frac{x}{a}\right)\sqrt{a^2 - x^2}\,dx = \frac{1}{4}a^2\left[\sin^{-1}\left(\frac{x}{a}\right)\right]^2 + \frac{1}{2}x\sqrt{a^2 - x^2}\,\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{8}\left(a^2 - 2x^2\right) + c.$$

Here a is a non-zero real constant and c is an arbitrary constant.

(ii) Show that $\int_0^p f(p-x)\,dx = \int_0^p f(x)\,dx$. Here $f(x)$ is a continuous function on the interval $[0, p]$.

Let $I = \int_0^a \frac{(a-x)^n}{(a-x)^n + x^n}\,dx$ and $J = \int_0^a \frac{x^n}{x^n + (a-x)^n}\,dx$.

Prove that $I = J$.

Obtain another relationship between I and J, and hence, find the value of I.

(iii) Evaluate $\int_0^2 (x+2)^3(x+5)\,dx$.

16. (i) $l_1 \equiv ax + by + c = 0$ and $l_2 \equiv px + qy + r = 0$ are two non-parallel straight lines. Show that for parameters λ and μ not equal to zero together, $\lambda l_1 + \mu l_2 = 0$ represents a straight line which passes through the intersection point of the straight lines $l_1 = 0$ and $l_2 = 0$.

The equations of the sides AB, BC, CD and DA of the quadrilateral $ABCD$ are respectively $2x + 10y - 9 = 0$, $8x + y - 1 = 0$, $x + 5y - 5 = 0$ and $7x - 4y - 15 = 0$. **Without finding** the coordinates of the points A and C, find the equation of the diagonal AC.

(ii) By using first principles, find the centre and the radius of the circle $x^2 + y^2 - 10x - 8y + 31 = 0$. Two tangents which are perpendicular to each other are drawn to this circle from a point on the x - axis. Show that there exists two such points.

17. (i) Express the *cosine rule* in the usual notation for any triangle ABC.
Hence, deduce that, $a = b \cos C + c \cos B$.
Show also that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

(ii) If $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$, show that $\cos(\alpha - \beta) = \frac{63}{65}$, and hence, find the value of $\sin(\alpha - \beta)$.

(iii) Express $\tan 3x$ in terms of $\tan x$ and show that the range of the function $f(x) = \tan 3x \cot x$ is $\mathbb{R} \setminus \left[\frac{1}{3}, 3\right)$.
Hence, solve the equation $\tan 3x - \tan x = 0$.

(iv) Write down the range of principal values of each of the inverse trigonometric functions $y = \sin^{-1} x$ and $y = \cos^{-1} x$.
Taking those ranges into account, prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

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G.C.E. (A. L.) Support Seminar - 2014

Combined Mathematics II

Three hours

⋆ In this question paper, g denotes the acceleration due to gravity.

Part A

Answer **all** the questions in the given space.

1. Let B be the highest point reached by a particle P which is projected vertically upwards under gravity from a point A on the ground with initial velocity u. If another particle Q falls from rest under gravity from the point B such that it meets P at the midpoint of AB, using the velocity-time graphs of both particles drawn on the same figure, prove that the motion of the particle Q must commence a time of $\left(\sqrt{2}-1\right)\frac{u}{g}$ before P is projected.

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2. A wedge of mass M has been placed on a smooth horizontal plane. A particle of mass m is placed on a smooth face of the wedge which is inclined at an angle α to the horizontal. When the system is released from rest, the particle moves a distance s along the inclined plane during the time that the wedge moves a distance d. **Without explicitly finding** the acceleration of the system, show that $(m+M)d = ms\cos\alpha$.

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3. A man projects a ball in a vertical plane under gravity, with horizontal and vertical velocity components of u and v respectively, from a point which is at a vertical height h from his feet. The ball collides perpendicularly with a smooth vertical wall which is at a horizontal distance d from his feet and bounces back to his feet. Show that $2ghe^2 = v^2(1-e^2)$. Here e is the coefficient of restitution between the ball and the wall.

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4. When a vehicle of mass M metric tons travels at a constant speed of u km h^{-1}, its engine power is H kW. If the vehicle is subject to a constant resistance of R N, show that $Ru = 3600H$.

Now, when the engine is switched off and the brakes are applied, the vehicle comes to rest after it travels a distance d km. By assuming that the resistance to the motion is constant throughout the motion, show that R', the retarding force of the brakes is given by $R'du = \frac{25}{648}Mu^3 - 3600Hd$.

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5. Q is the trisection point of the side AB of the triangle OAB which is closer to B. The point P lies on OQ and is such that $OP : OQ = 2 : 5$. The straight line AP produced, meets the side OB at R. Show that $\mathbf{OP} = \frac{2}{15}(\mathbf{a} + 2\mathbf{b})$, and write **AP** in terms of **a** and **b**. Here **a** and **b** are respectively the position vectors of A and B with respect to O.

Find the value that the scalar k can take such that $\mathbf{OA} + k\mathbf{AP}$ is independent of **a.**

By expressing the position vector **r** of R with respect to O in terms of **b**, show that $OR : OB = 4 : 13$.

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6. A uniform rod AB of length $2a$ and weight w, resting on a smooth peg at a point on the rod, is in equilibrium with end B above end A in a vertical plane perpendicular to a smooth vertical wall which is in contact with the end A of the rod. If the distance between the wall and the peg is d, and the rod is inclined at an angle θ to the horizontal, show that $\cos^3\theta = \frac{d}{a}$.

7. Let X and Y be two mutually exclusive events. If $P(X) \neq 0$ and $P(Y) \neq 0$, can X and Y be independent? Justify your answer.
A and B are two events such that $P(A) = \frac{1}{2}$, $P(A \mid B) = \frac{1}{4}$ and $P(B \mid A) = \frac{1}{3}$. Providing reasons, answer the following questions :
   (i) Are the events A and B independent?
   (ii) Are the events A and B mutually exclusive?

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8. The probabilities of the random events A and B occurring are $\frac{1}{2}$ and $\frac{1}{4}$ respectively. The probability of exactly one of these events occurring is $\frac{1}{3}$.

(i) Find the probability of both events occurring together.

(ii) Find the conditional probability of A occurring given that B has occurred.

9. Nine observations of which the mean and the mode are respectively 8 and 5 are given below.
5, 6, 13, 5, 10, 13, 3, x, y ; where $x < y$.
Find,

(i) the values of x and y separately.

(ii) the median of the 9 observations.

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10. The mean and the standard deviation of a population of 9 observations are 25 and 4 respectively. If the observations 15, 20 and 40 are added to it, calculate the mean and the standard deviation of the new population that is obtained.

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Part B

Answer **five** questions only.

11\. (a) At the same instance that an elevator starts its motion vertically upwards from rest with uniform acceleration, a ball is projected upwards from within the elevator with a vertical velocity u relative to the elevator. If the time of flight of the ball is T, draw a velocity-time graph for the motion of the ball relative to the elevator.

Using the velocity-time graph, show that if $u > \frac{1}{2}gT$, then the acceleration of the elevator is $\frac{1}{T}(2u - gT)$.

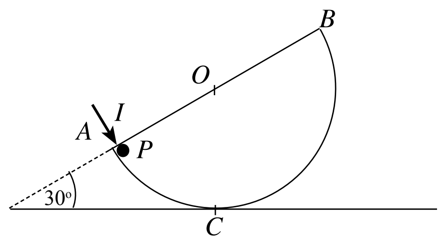
(b) A captain of a warship which is travelling towards the South at a speed of $u\,\text{km h}^{-1}$ observes an enemy ship at a distance of d km west of his ship, travelling at a speed of $u\sqrt{3}\ \text{km h}^{-1}$ in a direction 30° East from North.

(i) Find the velocity of the enemy ship.

(ii) Find the bearing of the enemy ship from the warship as well as the shortest distance between them when the two ships are closest to each other.

(iii) If the warship has a firing range of $0.9d$ km, show that the enemy ship is exposed to the danger of an attack by the warship for a period of $12\sqrt{2}\frac{d}{u}$ minutes.

12\. A smooth hemispherical bowl of centre O and internal radius a, is rigidly attached to a point C located on a fixed horizontal plane, such that its brim is inclined at an angle 30° to the horizontal as shown in the figure. The lowest point of the brim of the bowl is A and the highest point is B. The points A, B and C lie on the same vertical plane. A particle P of mass m is kept at rest at the point A and given an impulse I in a direction perpendicular to AB so that it moves in the plane ABC along the inner surface of the bowl.



(i) Find the initial velocity of the particle P.

(ii) Find the velocity of the particle P and the reaction on it from the bowl, when it is at a point between C and B such that the angle that OP makes with the downward vertical is α.

(iii) If the particle travels up to the point B, show that $I \geq \frac{m}{2}\sqrt{10ga}$.

(iv) If $I > \frac{m}{2}\sqrt{10ga}$, show that for the particle to pass through the point A in the subsequent motion, it is necessary that $I = \frac{m}{2}\sqrt{14ga}$.

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13. The end A of a light, smooth, elastic string AB with modulus of elasticity $3mg$ and natural length $3l$ has been rigidly attached to a point on a horizontal ceiling while a particle of mass m hangs from the other end B of the string. Assume that the ceiling is high enough so that the motion of the particle is not obstructed.

(i) Find the vertical distance from A to the equilibrium position of the particle.

Now, a small ring of mass m is kept at A such that the string passes through the ring, and the ring is released from rest. The ring moves downwards along the string.

(ii) Show that the velocity of the ring just before it collides with the particle is $2\sqrt{2gl}$.

(iii) After the ring collides with the particle, if both become a single body, show that the initial velocity of the combined body is $\sqrt{2gl}$.

(iv) Show that the combined body moves with simple harmonic motion and find its angular velocity and the period of oscillation.

(v) Show that the ceiling should be at a minimum height of $(5+\sqrt{5})l$ above the floor in order for the simple harmonic motion of the combined body to continue without touching the floor.

14. (a) **a** and **b** are two non-zero vectors which are **not** parallel to each other. If λ and μ are two given scalars, show that $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{0}$ if and only if $\lambda = \mu = 0$.

The side BC of the parallelogram $OACB$ has been produced to D such that $BD = 3BC$.
By taking $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, express **OD** in terms of **a** and **b**.

Find the constants λ and μ such that $\mathbf{OE} = \lambda\,\mathbf{OD}$ and $\mathbf{AE} = \mu\mathbf{AC}$, where E is the intersection point of OD and AC.

If **i** and **j** are unit vectors which are perpendicular to each other, and the vectors **a** and **b** have been expressed as $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, write down in terms of x_1, y_1, x_2 and y_2, the condition that should be satisfied for $OACB$ to be a rhombus.

(b) The mid-points of the sides AB, BC, CD and DA of the rectangle $ABCD$ are respectively P, Q, R and S, where $AB = 6a$ and $BC = 2\sqrt{3}a$. Six forces of magnitude 15 N, λ N, 5 N, 10 N, μ N and $30\sqrt{3}$ N act along the sides PQ, QR, RS, SP, AD and CD respectively in the direction indicated by the order of the letters.
Show that,

(i) this system of forces **cannot** be in equilibrium,

(ii) if this system of forces reduces to a couple, then $\lambda = -40$ and $\mu = 20$,

(iii) if the system reduces to a force of 10 N acting along the direction AD, then $\lambda = -40$ and $\mu = 30$.

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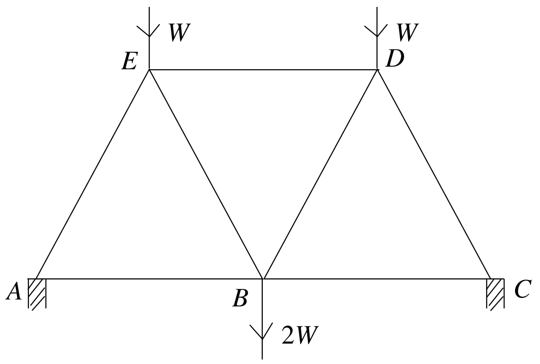
15. (a) The weights of three uniform rods AB, BC and CD, each of length $4a$ are λW, W and λW respectively. The rods are smoothly jointed at B and C. The ends A and D are freely hinged to two fixed points in the same horizontal level, a distance $8a$ apart from each other, such that the system is in equilibrium in a vertical plane with BC lying below AD.

(i) Find the horizontal and vertical components of the reaction at A.

(ii) Find the horizontal and vertical components of the reaction on the rod BC at the point B.

(iii) If the lines of action of the reactions on the rod BC at B and C meet at a point a distance $\frac{\sqrt{3}}{2}a$ vertically below BC, find the value of λ.

(b)



A framework made of seven light rods of equal length is shown in the diagram. It is kept on two supports at A and C, in equilibrium in a vertical plane, and the loads $2W$, W and W are applied on the framework at B, D and E respectively.

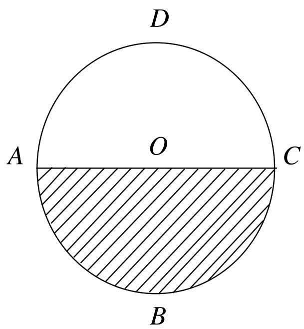
(i) Find the reaction of the support at C on the framework.

(ii) Draw a stress diagram for the framework using Bow's notation and hence find the stresses in the rods in terms of W. Indicate whether each stress is a tension or a thrust.

16. (a) Show that the centre of mass of a thin uniform wire, the shape of an arc of a circle, is at a distance $\frac{r\sin\alpha}{\alpha}$ from the centre of the arc. Here r is the radius of the arc and 2α is the angle subtended by the arc at the centre.

Hence find the centre of mass of a thin semi-circular lamina.

A composite body has been made as shown in the figure, using a thin semi-circular lamina ABC of radius r and a thin semi-circular wire ADC. The centre of the circle is O. If the surface density of the material with which the lamina has been made is σ, and the linear density of the material with which the wire has been made is k, find the distance from O to the centre of mass of the composite body. When this composite body is freely suspended from the point A, find the angle that AB makes with the vertical. Also, if O lies vertically below A when the composite body is suspended in the above manner, find the ratio $k : \sigma$.



(b) A rod is in limiting equilibrium in a vertical plane, with one end of the rod on a rough horizontal plane and another point on the rod resting on a smooth fixed peg. If the reaction of the rough horizontal plane on the rod is equal in magnitude to the weight of the rod, show that the inclination of the rod to the vertical is equal to half the angle of friction.

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17.(a) A house-wife goes every Sunday to one of the following, namely the nearby fish outlet, Sathosa or the market with probabilities of $\frac{2}{5}, \frac{2}{5}$ and $\frac{1}{5}$ respectively to purchase the type of fish she likes most for her weekly consumption. She is able to purchase this type of fish from each of these places with probabilities of $\frac{1}{5}, \frac{1}{2}$ and $\frac{3}{5}$ respectively.

(i) Find the probability of her being able to purchase the type of fish she likes most on a particular Sunday.

(ii) If on a particular Sunday she was unable to purchase the type of fish she likes most, this would most likely be because she went to which of the three places?
Justify your answer.

(iii) Assuming that the events occurring on the different Sundays are independent of each other, find the probability of her being able to purchase the type of fish she likes most on at least two Sundays out of any three consecutive Sundays.

(iv) Her neighbour too goes every Sunday to the same places to buy fish with probabilities of $\frac{2}{5}, \frac{2}{5}$ and $\frac{1}{5}$ respectively. If the two of them make decisions independently, find the probability of both going to the same place to buy fish on a particular Sunday.

(b) Define the mean $\bar{x}$ and the standard deviation s_x of the group of data x_i ; $i = 1, 2, 3, ..., n$. Considering the transformation $u_i = a + bx_i$, prove that $\bar{u} = a + b\bar{x}$ and $s_u^2 = b^2 s_x^2$. Here $\bar{u}$ and s_u are respectively the mean and the standard deviation of the group of values u_i.

x_i ; $i = 1, 2, 3, ..., n_1$ and y_i ; $i = 1, 2, 3, ..., n_2$ are two groups of observations. $\bar{x} = 35$ and $s_x = 4$. The group of values x_i are transformed to the group of values u_i using the transformation $u_i = 70 + 3x_i$. Calculate $\bar{u}$ and s_u.

If it is required to transform the group of values y_i to the same group of values u_i using the transformation $u_i = a + by_i$, and if it is given that $\bar{y} = 19$ and $s_y = \frac{12}{5}$, find a and b such that $a, b > 0$.

If the x value 55 and the y value 32 are transformed using the relevant transformation, determine for which of these two observations a greater u value is obtained.

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