7 Algebraic Fractions

By studying this lesson, you will understand

how algebraic fractions are multiplied and divided.

Do the following exercise to revise what you have learned before on adding and subtracting algebraic fractions.

Review Exercise

Simplify.

a. $\frac{a}{5}+\frac{2a}{5}$ **b.** $\frac{8}{x}-\frac{3}{x}$ **c.** $\frac{7}{3m}+\frac{3}{4m}-\frac{8}{m}$

d. $\frac{9}{x+2}+\frac{1}{x}$ **e.** $\frac{1}{m+2}-\frac{2}{m+3}$ **f.** $\frac{a+3}{a^2-4}+\frac{1}{a+2}$

g. $\frac{2}{x^2-x-2}-\frac{1}{x^2-1}$ **h.** $\frac{1}{x^2-9x+20}-\frac{1}{x^2-11x+30}$

7.1 Multiplying algebraic fractions

Two algebraic fractions can be multiplied in the same way that two numerical fractions are multiplied. Let us consider the following example.

$\frac{x}{2}\times\frac{x}{3}$

What we mean by performing the multiplication is to express this product as a single fraction.

To perform this multiplication, we multiply the numerators and denominators of the two fractions separately, and obtain a single fraction. That is,

$$\frac{x}{2}\times\frac{x}{3} = \frac{x\times x}{2\times 3}$$

$$= \frac{x^2}{6}$$

If the terms in the numerator and the denominator can be further simplified, by doing the simplification we can express the answer in the simplest form. These simplifications can be done either before multiplying the fractions or after multiplying the fractions. Let us now consider multiplying two fractions where such simplifications are possible.

Consider $\dfrac{8}{a} \times \dfrac{3}{2b}$.

Here, we can cancel the common factor 2, of the numerator 8 of the first fraction and the denominator $2b$ of the second fraction. We perform this simplification as follows.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{{}^{4}\cancel{8}}{a} \times \frac{3}{{}_{1}\cancel{2}b}$$

Now, by multiplying the expressions in the numerators and denominators of the two fractions separately, we get a single fraction as given below.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{4 \times 3}{a \times b}$$

$$= \underline{\underline{\frac{12}{ab}}} \; .$$

We can also cancel the common factors after multiplying the fractions. Consider the following example.

$$\frac{3}{2a} \times \frac{2b}{3} = \frac{6b}{6a}$$

$$= \frac{b}{a}$$

However, by cancelling the common factors before doing this the multiplication, you can minimise long multiplications and divisions. Therefore, doing this is encouraged.

Observe how the following algebraic expressions are simplified.

Example 1

$$\frac{x}{y} \times \frac{4}{5x}$$

$= \dfrac{{}^{1}\cancel{x}}{y} \times \dfrac{4}{5\cancel{x}_{1}}$ (Dividing by the common factor x)

$$= \frac{1 \times 4}{y \times 5}$$

$$= \underline{\underline{\frac{4}{5y}}}$$

When multiplying fractions with algebraic expressions in the numerator and the denominator, first factorise the expressions. This is done to cancel the common factors if there are any. Consider the following example.

Example 2

Simplify $\dfrac{2}{x+3} \times \dfrac{x^2+3x}{5}$

$$\frac{2}{x+3} \times \frac{x^2+3x}{5} = \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Factorise } x^2+3x \text{)}$$

$$= \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Divide by the common factor } x+3\text{)}$$

$$= \frac{2x}{5}$$

Let us now consider a slightly complex example.

Example 3

Simplify $\dfrac{a^2-9}{5a} \times \dfrac{2a-4}{a^2+a-6}$

$$\frac{a^2-9}{5a} \times \frac{2a-4}{a^2+a-6} = \frac{a^2-3^2}{5a} \times \frac{2\,(a-2)}{(a+3)\,(a-2)}$$

$$= \frac{(a-3)\,(a+3)}{5a} \times \frac{2(a-2)}{(a+3)\,(a-2)}$$

$$= \frac{2\,(a-3)}{5a}$$

because $a^2 - 3^2 = (a-3)\,(a+3)$

because $a^2 + a - 6 = (a+3)\,(a-2)$

Exercise 7.1

Multiply the following algebraic fractions.

a. $\dfrac{6}{x} \times \dfrac{2}{3x}$

b. $\dfrac{x}{5} \times \dfrac{3}{xy}$

c. $\dfrac{2a}{15} \times \dfrac{5}{9}$

d. $\dfrac{4m}{5n} \times \dfrac{3}{2m}$

e. $\dfrac{x+1}{8} \times \dfrac{2x}{x+1}$

f. $\dfrac{3a-6}{3a} \times \dfrac{1}{a-2}$

g. $\dfrac{x^2}{2y+5} \times \dfrac{4y+10}{3x}$

h. $\dfrac{m^2-4}{m+1} \times \dfrac{m^2+2m+1}{m+2}$

i. $\dfrac{x^2-5x+6}{x^2-1} \times \dfrac{x^2-2x-3}{x^2-9}$

j. $\dfrac{a^2-b^2}{a^2-2ab+b^2} \times \dfrac{2a-2b}{a^2+ab}$

7.2 Dividing an algebraic fraction by another

Recall how you obtained the answer when dividing one fraction by another fraction. You multiplied the first fraction by the reciprocal of the second fraction. Similarly, when dividing an algebraic fraction by another algebraic fraction, we can instead multiply the first by the reciprocal of the second.

Before we study how algebraic fractions are divided, let us consider the reciprocal of an algebraic fraction.

Reciprocal of an algebraic fraction

Recall the facts we have leaned regarding the reciprocal of a number. If the product of two numbers is 1, then each number is the reciprocal or the multiplicative inverse of the other number.

Because $2 \times \frac{1}{2} = 1$, reciprocal of 2 is $\frac{1}{2}$ and reciprocal of $\frac{1}{2}$ is 2.

Because $\frac{1}{3} \times 3 = 1$, reciprocal of $\frac{1}{3}$ is 3 and reciprocal of 3 is $\frac{1}{3}$.

Because $\frac{4}{5} \times \frac{5}{4} = 1$, reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$ and reciprocal of $\frac{5}{4}$ is $\frac{4}{5}$.

The reciprocal of an algebraic fraction is also described similarly. That is, if the product of two algebraic fractions is 1, then each algebraic fraction is the reciprocal of the other.

Let us multiply the two algebraic fractions $\frac{5}{x}$ and $\frac{x}{5}$.

$\frac{5}{x} \times \frac{x}{5} = \frac{1}{1} = 1.$

Therefore, $\frac{5}{x}$ is the reciprocal of $\frac{x}{5}$ and $\frac{x}{5}$ is the reciprocal of $\frac{5}{x}$.

Similarly, because

$\frac{x+1}{y} \times \frac{y}{x+1} = 1,$

$\frac{x+1}{y}$ is the reciprocal of $\frac{y}{x+1}$ and $\frac{y}{x+1}$ is the reciprocal of $\frac{x+1}{y}$.

Now it should be clear that, the reciprocal of an algebraic fraction can be obtained by simply interchanging the numerator and denominator, as you have done with numerical fractions.

Observe the following algebraic fractions and their reciprocals.

algebraic fraction	reciprocal
$\frac{m}{4}$	$\frac{4}{m}$
$\frac{a}{a+2}$	$\frac{a+2}{a}$
$\frac{x-3}{x^2+5x+6}$	$\frac{x^2+5x+6}{x-3}$

Let us now consider how to divid an algebraic fraction by another.

Example 1

Simplify $\frac{3}{x} \div \frac{4y}{x}$

$\frac{3}{x} \div \frac{4y}{x} = \frac{3}{x} \times \frac{x}{4y}$ (Instead of dividing by $\frac{4y}{x}$ we multiply by its reciprocal $\frac{x}{4y}$)

$= \frac{3}{\cancel{x}} \times \frac{\cancel{x}}{4y}$ (Dividing by the common factor x)

$= \underline{\underline{\frac{3}{4y}}}$ (Multiplying the numerators and denominators separately)

Let us consider a few more examples.

Example 2

Simplify $\frac{a}{b} \div \frac{ab}{4}$

$\frac{a}{b} \div \frac{ab}{4} = \frac{a}{b} \times \frac{4}{ab}$ (Multiplying by the reciprocal)

$= \frac{\cancel{a}}{b} \times \frac{4}{\cancel{a}b}$ (Cancelling a)

$= \underline{\underline{\frac{4}{b^2}}}$

When there are algebraic expressions in both the numerator and the denominator, we can first factor the expressions, so that the common factors can be easily found and cancelled before simplifying.

Look at the following examples.

Example 3

Simplify $\dfrac{3x}{x^2 + 2x} \div \dfrac{5x}{x^2 - 4}$

$$\frac{3x}{x^2 + 2x} \div \frac{5x}{x^2 - 4} = \frac{3x}{x^2 + 2x} \times \frac{x^2 - 4}{5x}$$ (Multiplying by the reciprocal)

$$= \frac{3x}{x\,(x + 2)} \times \frac{(x - 2)\,(x + 2)}{5x}$$ (Factoring the expressions and dividing by the common factors)

$$= \underline{\underline{\frac{3\,(x - 2)}{5x}}}$$

Example 4

Simplify $\dfrac{x^2 + 3x - 10}{x} \div \dfrac{x^2 - 25}{x^2 - 5x}$

$$\frac{x^2 + 3x - 10}{x} \div \frac{x^2 - 25}{x^2 - 5x} = \frac{x^2 + 3x - 10}{x} \times \frac{x^2 - 5x}{x^2 - 25}$$

$$= \frac{(x + 5)\,(x - 2)}{x} \times \frac{x\,(x - 5)}{(x - 5)\,(x + 5)}$$

$$= \frac{x - 2}{1}$$

$$= \underline{\underline{x - 2}}$$

Exercise 7.2

Simplify the following algebraic fractions.

a. $\frac{5}{x} \times \frac{10}{x}$

b. $\frac{m}{3n} \div \frac{m}{2n^2}$

c. $\frac{x+1}{y} \div \frac{2(x+1)}{x}$

d. $\frac{2a-4}{2a} \div \frac{a-2}{3}$

e. $\frac{x^2+4x}{3y} \div \frac{x^2-16}{12y^2}$

f. $\frac{p^2+pq}{p^2-pr} \div \frac{p^2-q^2}{p^2-r^2}$

g. $\frac{m^2-4}{m+1} \div \frac{m+2}{m^2+2m+1}$

h. $\frac{x^2y^2+3xy}{4x^2-1} \div \frac{xy+3}{2x+1}$

i. $\frac{a^2-5a}{a^2-4a-5} \div \frac{a^2-a-2}{a^2+2a+1}$

j. $\frac{x^2-8x}{x^2-4x-5} \times \frac{x^2+2x+1}{x^3-8x^2} \div \frac{x^2+2x-3}{x-5}$