# 6 Binomial Expressions

**By studying this lesson, you will be able to**

expand the cube (third power) of a binomial expression.

You have learned in earlier lessons that, for a binomial expression of the form $x + y$, its square is denoted by $(x + y)^2$, and that what it means is $(x + y)(x + y)$, and that when the product is expanded, the expression $x^2 + 2xy + y^2$ is obtained. Moreover, recall that $x^2 - 2xy + y^2$ is obtained when $(x - y)^2$ is expanded.

Do the following exercise to recall what you have learned about the expansion of squares of binomial expressions.

**Review Exercise**

**1.** Fill in the blanks.

**a.** $(a + b)^2 = a^2 + 2ab + .....$

**b.** $(a - b)^2 = .... - 2ab + b^2$

**c.** $(x + 2)^2 = x^2 + 4x + .....$

**d.** $(y + 3)^2 = y^2 + .....\ + 9$

**e.** $(a - 5)^2 = .....\ - 10a + 25$

**f.** $(b - 1)^2 = b^2\ .....\ +\ .....$

**g.** $(4 + x)^2 = 16 + .....\ .....$

**h.** $(7 - t)^2 = 49\ .....\ + t^2$

**i.** $(2x + 1)^2 = 4x^2\ .....\ +\ 1$

**j.** $(3b - 2)^2 = ......\ - 12b\ ......$

**2.** Expand.

**a.** $(2m + 3)^2$ **b.** $(3x - 1)^2$ **c.** $(5+2x)^2$

**d.** $(2a + 3b)^2$ **e.** $(3m - 2n)^2$ **f.** $(2x + 5y)^2$

**3.** Evaluate the following squares, by writing each as a square of a binomial expression.

**a.** $32^2$ **b.** $103^2$ **c.** $18^2$ **d.** $99^2$

## 6.1 Cube of a binomial expression

The cube of the binomial expression $a + b$, is $(a + b)^3$. That is, the third power of $(a + b)$. Note that this is the same as multiplying $(a + b)^2$ again by $(a + b)$.

Carefully observe how the following expressions, involving a power of 3, are written.

$$3^3 = 3 \times 3^2 = 3 \times 3 \times 3 = 27$$
$$x^3 = x \times x^2 = x \times x \times x$$
$$(2x)^3 = (2x) \times (2x)^2 = (2x) \times (2x) \times (2x) = 8x^3$$

In a similar way, we can write

$$(x + 1)^3 = (x + 1)(x + 1)^2 = (x + 1)\,(x + 1)\,(x + 1)$$
$$(a - 2)^3 = (a - 2)(a - 2)^2 = (a - 2)\,(a - 2)\,(a - 2)$$
$$(3 + m)^3 = (3 + m)(3 + m)^2 = (3 + m)\,(3 + m)\,(3 + m)$$

The cube of a binomial expression can be expanded in a way similar to how the square of a binomial expression was expanded. It is illustrated in the following example.

**Example 1**

$$(x + y)^3 = (x + y)\,(x + y)^2$$
$$= (x + y)\quad(x^2 + 2xy + y^2)$$
$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$
$$= \underline{\underline{x^3 + 3x^2y + 3xy^2 + y^3}}$$

Accordingly, let us remember the following pattern as a formula for the expansion of the cube of the binomial expression $(x + y)$.

$$\boxed{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$$

cube of the first term

three times the product of the square of the first term and the second term.

three times the product of the first term and the square of the second term

cube of the second term

According to this, we can write

$$(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$$

Similarly, we can write $(a + 2)^3 = a^3 + 3a^2 \times 2 + 3a \times 2^2 + 2^3$, and this can be further simplified as ,

$a^3 + 6a^2 + 12a + 8$

Now let us consider how the expansion of $(x - y)^3$ is obtained by taking products.

$$
\begin{aligned}
(x - y)^3 &= (x - y)\,(x - y)^2 \\
&= (x - y)\;(x^2 - 2xy + y^2) \\
&= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3 \\
&= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}
\end{aligned}
$$

Now, let us consider how we can obtain the expansion of $(x - y)^3$, using another method.

First, note that we can write $x - y$ as $x + (-y)$. Therefore, we can treat $(x - y)^3$ as an expression of the initial form, by writing it as $\{x + (-y)\}^3$. Let us now consider the expansion of this cube.

$$
\begin{aligned}
(x - y)^3 = \{x + (-y)\}^3 &= x^3 + 3 \times x^2 \times (-y) + 3 \times x \times (-y)^2 + (-y)^3 \\
&= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}
\end{aligned}
$$

Note that we have used the properties $(-y)^2 = y^2$ and $(-y)^3 = -y^3$ in the above simplification.

According to this, we can also write

$$(m - n)^3 = m^3 - 3m^2n + 3mn^2 - n^3$$
$$(p - q)^3 = p^3 - 3p^2q + 3pq^2 - q^3$$

Either method can be used to obtain the expansion of $(x - y)^3$. You may use any method which is easy for you.

Let us now consider how the cube of a binomial expression, involving numbers as well, is expanded.

**Example 2**

$$
\begin{aligned}
(x + 5)^3 &= x^3 + 3 \times x^2 \times 5 + 3 \times x \times 5^2 + 5^3 \\
&= \underline{\underline{x^3 + 15x^2 + 75x + 125}}
\end{aligned}
$$

**Example 3**

$$
\begin{aligned}
(1 + x)^3 &= 1^3 + 3 \times 1^2 \times x + 3 \times 1 \times x^2 + x^3 \\
&= \underline{\underline{1 + 3x + 3x^2 + x^3}}
\end{aligned}
$$

**Example 4**

$$(y-4)^3 = y^3 + 3 \times y^2 \times (-4) + 3 \times y \times (-4)^2 + (-4)^3$$
$$= \underline{\underline{y^3 - 12y^2 + 48y - 64}}$$

or

$$(y-4)^3 = y^3 - 3 \times y^2 \times 4 + 3 \times y \times 4^2 - 4^3$$
$$= \underline{\underline{y^3 - 12y^2 + 48y - 64}}$$

**Example 5**

$$(5-a)^3 = 5^3 + 3 \times 5^2 \times (-a) + 3 \times 5 \times (-a)^2 + (-a)^3$$
$$= \underline{\underline{125 - 75a + 15a^2 - a^3}}$$

or

$$(5-a)^3 = 5^3 - 3 \times 5^2 \times a + 3 \times 5 \times a^2 - a^3$$
$$= \underline{\underline{125 - 75a + 15a^2 - a^3}}$$

**Example 6**

$$(-2+a)^3 = (-2)^3 + 3 \times (-2)^2 \times a + 3 \times (-2) \times a^2 + a^3$$
$$= \underline{\underline{-8 + 12a - 6a^2 + a^3}}$$

**Example 7**

$$(-3-b)^3 = (-3)^3 + 3 \times (-3)^2 \times (-b) + 3 \times (-3) \times (-b)^2 + (-b)^3$$
$$= \underline{\underline{-27 - 27b - 9b^2 - b^3}}$$

or

$$\left[-1\,(3+b)\right]^3 = (-1)^3\ (3+b)^3$$
$$= -1\,(3^3 + 3 \times 3^2 \times b + 3 \times 3 \times b^2 + b^3)$$
$$= -1\,(27 + 27b + 9b^2 + b^3)$$
$$= \underline{\underline{-27 - 27b - 9b^2 - b^3}}$$

**Example 8**

Write the expansion of $(x-3)^3$ and verify that $(4-3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

$(x-3)^3 = x^3 - 3^2 \times x^2 + 3^3 \times x - 3^3$

Substituting $x = 4$

Left s. $= (4-3)^3$
$= 1$

Right s. $= 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$
$= 1$

Left s. = Right s.

Therefore $(4-3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

**Exercise 6.1**

**1.** Fill in the blanks using suitable algebraic terms, symbols (+ or –) or numbers.

**a.** $(x+3)^3 = x^3 + 3 \times x^2 \times 3 + 3 \times x \times 3^2 + 3^3 = x^3 + \square + \square + 27$

**b.** $(y+2)^3 = y^3 + 3 \times \square \times \square + 3 \times \square \times \square + 2^3 = y^3 + 6y^2 + \square + \square$

**c.** $(a-5)^3 = a^3 + 3 \times a^2 \times (-5) + 3 \times a \times (-5)^2 + (-5)^3 = a^3 - \square + \square - 125$

**d.** $(3+t)^3 = \square + 3 \times \square \times \square + 3 \times \square \times \square + \square = \square + 27t + \square + t^3$

**e.** $(x-2)^3 = x^3 \square\, 3 \times \square \times \square + 3 \times \square \times \square + (-2)^3 = x^3 \square \square + 12x - \square$

**2.** Expand.

**a.** $(m+2)^3$ **b.** $(x+4)^3$ **c.** $(b-2)^3$ **d.** $(t-10)^3$

**e.** $(5+p)^3$ **f.** $(6+k)^3$ **g.** $(1+b)^3$ **h.** $(4-x)^3$

**i.** $(2-p)^3$ **j.** $(9-t)^3$ **k.** $(-m+3)^3$ **l.** $(-5-y)^3$

**m.** $(ab+c)^3$ **n.** $(2x+3y)^3$ **o.** $(3x+4y)^3$ **p.** $(2a-5b)^3$

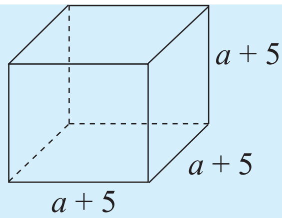
**3.** Write as a cube of a binomial expression.

**a.** $a^3 + 3a^2b + 3ab^2 + b^3$ **b.** $c^3 - 3c^2d + 3cd^2 - d^3$

**c.** $x^3 + 6x^2 + 12x + 8$ **d.** $y^3 - 18y^2 + 108y - 216$

**e.** $1 + 3x + 3x^2 + x^3$ **f.** $64 - 48x + 12x^2 - x^3$

**4.** Shown in the diagram is a cube with the length of each side $(a + 5)$ units. Write an expression for the volume of the cube and expand it.




**5.** Expand $(x + 3)^3$ , and verify the result for the following cases.

(i) $x = 2$
(ii) $x = 4$

.

**6.** Use the knowledge on cubes of binomial expressions to evaluate the following numerical expressions.

(i) $64 - 3 \times 16 \times 3 + 3 \times 4 \times 9 - 27$
(ii) $216 - 3 \times 36 \times 5 + 3 \times 6 \times 25 - 125$

**7.** Find the value of each of the following, by writing each as a cube of a binomial expression.

***a.*** $21^3$ **b.** $102^3$ **c.** $17^3$ **d.** $98^3$

**8.** Find the volume of a cube, with each side $2a - 5$ cm, in terms of $a$.

**9.** Write $x^3 - 3x^2y + 3xy^2 - y^3$ as a cube and use it to find the value of $25^3 - 3\times 25^2\times 23 + 3 \times 25 \times 23^2 - 23^3$.