2 Indices and Logarithms I

By studying this lesson, you will be able to

- simplify expressions involving powers and roots and
- solve equations

using the laws of indices and logarithms.

Indices

Do the following exercise to revise what you have learned so far about indices and logarithms.

Review Exercise

1. Simplify and find the value.

a. $2^2\times 2^3$ **b.** $(2^4)^2$ **c.** 3^{-2}

d. $\frac{5^3\times 5^2}{5^5}$ **e.** $\frac{3^5\times 3^2}{3^6}$ **f.** $(5^2)^2 \div 5^3$

g. $\frac{(2^2)^3\times 2^4}{2^8}$ **h.** $\frac{5^{-3}\times 5^2}{5^0}$ **i.** $(5^2)^{-2}\times 5\times 3^0$

2. Simplify.

a. $a^2\times a^3\times a$ **b.** $a^5\times a\times a^0$ **c.** $(a^2)^3$

d. $(x^2)^3\times x^2$ **e.** $(xy)^2\times x^0$ **f.** $(2x^2)^3$

g. $\frac{2pq\times 3p}{6p^2}$ **h.** $2x^{-2}\times 5xy$ **i.** $\frac{(3a)^{-2}\times 4a^2b^2}{2ab}$

3. Simplify.

a. $\lg 25 + \lg 4$ **b.** $\log_2 8 - \log_2 4$

c. $\log_5 50 + \log_5 2 - \log_5 4$ **d.** $\log_a 5 + \log_a 4 - \log_a 2$

e. $\log_x 4 + \log_x 12 - \log_x 3$ **f.** $\log_p a + \log_p b - \log_p c$

4. Solve the following equations.

a. $\log_5 x = \log_5 4 + \log_5 2$ **b.** $\log_5 4 - \log_5 2 = \log_5 x$

c. $\log_a 2 + \log_a x = \log_a 10$ **d.** $\log_3 x + \log_3 10 = \log_3 5 + \log_3 6 - \log_3 2$

e. $\lg 5 - \lg x + \lg 8 = \lg 4$ **f.** $\log_x 12 - \log_5 4 = \log_5 3$

2.1 Fractional Indices of a Power

Square root of 4 can be written either using the radical symbol (square root symbol) as $\sqrt{4}$ or using powers as $4^{\frac{1}{2}}$.

Therefore, it is clear that $\sqrt{4} = 4^{\frac{1}{2}}$.

Let us consider another example, similar to the above. As $2 = 2^1$,

$$\begin{aligned} 2 \times 2 \times 2 &= 2^1 \times 2^1 \times 2^1 \\ &= 2^3 \\ &= 8 \end{aligned}$$

Third power of 2 is 8. Thus, the cube root of 8 is 2. This can be denoted symbolically as,

$$\sqrt[3]{8} = 2 \text{ or } 8^{\frac{1}{3}} = 2.$$

Therefore it is clear that $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Futhermore, if a is a positive real number, then

$$\sqrt{a} = a^{\frac{1}{2}},$$
$$\sqrt[3]{a} = a^{\frac{1}{3}} \text{ and}$$
$$\sqrt[4]{a} = a^{\frac{1}{4}}.$$

Thus, the general relationship between the radical symbol and the exponent (index) of a power can be expressed as follows.

$$\boxed{\sqrt[n]{a} = a^{\frac{1}{n}}}$$

The following examples demonstrate how the above relationship can be used to simplify expressions involving powers.

Example 1

1. Find the value.

(i) $\sqrt[3]{27}$ (ii) $(\sqrt{25})^{-2}$ (iii) $\sqrt[3]{3\frac{3}{8}}$

(i) $\sqrt[3]{27} = 27^{\frac{1}{3}}$

$= (3^3)^{\frac{1}{3}}$

$= 3^{3 \times \frac{1}{3}}$

$= \underline{\underline{3}}$

(ii) $(\sqrt{25})^{-2} = (25^{\frac{1}{2}})^{-2}$

$= \{(5^2)^{\frac{1}{2}}\}^{-2}$

$= (5^{2 \times \frac{1}{2}})^{-2}$

$= 5^{-2}$

$= \frac{1}{5^2}$

$= \underline{\underline{\frac{1}{25}}}$

(iii) $\sqrt[3]{3\frac{3}{8}} = \sqrt[3]{\frac{27}{8}}$

$= \left(\frac{27}{8}\right)^{\frac{1}{3}}$

$= \frac{(3^3)^{\frac{1}{3}}}{(2^3)^{\frac{1}{3}}}$

$= \frac{3^{3 \times \frac{1}{3}}}{2^{3 \times \frac{1}{3}}}$

$= \frac{3}{2}$

$= \underline{\underline{1\frac{1}{2}}}$

The following examples further investigate how the laws of indices are used to simplify algebraic expressions involving powers.

Example 2

Simplify and express the answer with positive exponents (indices).

(i) $(\sqrt{x})^3$ (ii) $(\sqrt[3]{a})^{-\frac{1}{2}}$ (iii) $\frac{1}{\sqrt{x^{-3}}}$

(i) $(\sqrt{x})^3 = (x^{\frac{1}{2}})^3$

$= x^{\frac{1}{2} \times 3}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

(ii) $(\sqrt[3]{a})^{-\frac{1}{2}} = (a^{\frac{1}{3}})^{-\frac{1}{2}}$

$= a^{\frac{1}{3} \times -\frac{1}{2}}$

$= a^{-\frac{1}{6}}$

$= \underline{\underline{\frac{1}{a^{\frac{1}{6}}}}}$

(iii) $\frac{1}{\sqrt{x^{-3}}} = \frac{1}{(x^{-3})^{\frac{1}{2}}}$

$= \frac{1}{x^{-3 \times \frac{1}{2}}}$

$= \frac{1}{x^{-\frac{3}{2}}}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

Example 3

Find the value. (i) $\left(\frac{27}{64}\right)^{\frac{2}{3}}$ (ii) $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

(i)
$$\left(\frac{27}{64}\right)^{\frac{2}{3}} = \left(\frac{3^3}{4^3}\right)^{\frac{2}{3}}$$
$$= \left[\left(\frac{3}{4}\right)^3\right]^{\frac{2}{3}}$$
$$= \left(\frac{3}{4}\right)^{3\times\frac{2}{3}}$$
$$= \left(\frac{3}{4}\right)^2$$
$$= \underline{\underline{\frac{9}{16}}}$$

(ii)
$$\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{-\frac{3}{4}}$$
$$= \left(\frac{2}{3}\right)^{4\times -\frac{3}{4}}$$
$$= \left(\frac{2}{3}\right)^{-3}$$
$$= \left(\frac{3}{2}\right)^3$$
$$= \frac{27}{8}$$
$$= \underline{\underline{3\frac{3}{8}}}$$

Let us now consider a slightly complex example: $\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \sqrt[5]{32}^{\,3} \times 3^0$

$$\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \left(\sqrt[5]{32}\right)^3 \times 3^0 = \left(\frac{5^3}{2^6}\right)^{-\frac{1}{3}} \times \left(32^{\frac{1}{5}}\right)^3 \times 1$$
$$= \left(\frac{2^6}{5^3}\right)^{\frac{1}{3}} \times \left(2^{5\times\frac{1}{5}}\right)^3$$
$$= \frac{2^{6\times\frac{1}{3}}}{5^{3\times\frac{1}{3}}} \times 2^3$$
$$= \frac{2^2}{5} \times 2^3$$
$$= \frac{2^5}{5}$$
$$= \frac{32}{5}$$
$$= \underline{\underline{6\frac{2}{5}}}$$

Example 4

Simplify: $\dfrac{\sqrt[3]{343x^{\frac{3}{2}}}}{x}$

$$\frac{\sqrt[3]{343x^{\frac{3}{2}}}}{x} = \left(343x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x$$

$$= 343^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x$$

$$= (7^3)^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x$$

$$= 7^1 \times x^{\frac{1}{2}} \div x$$

$$= 7 \times x^{\frac{1}{2} - 1}$$

$$= 7 \times x^{-\frac{1}{2}}$$

$$= \frac{7}{x^{\frac{1}{2}}}$$

Exercise 2.1

1. Express the following using the radical symbol.

a. $p^{\frac{1}{3}}$ **b.** $a^{\frac{2}{3}}$ **c.** $x^{-\frac{2}{3}}$

d. $m^{\frac{4}{5}}$ **e.** $y^{-\frac{3}{4}}$ **f.** $x^{-\frac{5}{3}}$

2. Write using positive exponents (indices).

a. $\sqrt{m^{-1}}$ **b.** $\sqrt[3]{x^{-1}}$ **c.** $\sqrt[5]{p^{-2}}$ **d.** $(\sqrt{a})^{-3}$ **e.** $\sqrt[4]{x^{-3}}$

f. $(\sqrt[3]{p})^{-5}$ **g.** $\dfrac{1}{\sqrt{x^{-3}}}$ **h.** $\dfrac{1}{\sqrt[3]{a^{-2}}}$ **i.** $2\sqrt[3]{x^{-2}}$ **j.** $\dfrac{1}{3\sqrt{a^{-5}}}$

3. Find the value.

a. $\sqrt{25}$ **b.** $\sqrt[4]{16}$ **c.** $(\sqrt{4})^5$

d. $(\sqrt[3]{27})^2$ **e.** $\sqrt[4]{81^3}$ **f.** $\sqrt[3]{1000^2}$

g. $\left(\frac{27}{125}\right)^{\frac{2}{3}}$ **h.** $\left(\frac{81}{10000}\right)^{\frac{3}{4}}$ **i.** $\left(\frac{1}{64}\right)^{-\frac{5}{6}}$

j. $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$ **k.** $(0.81)^{\frac{3}{2}}$ **l.** $(0.125)^{-\frac{2}{3}}$

m. $\left(\frac{4}{25}\right)^{\frac{1}{2}} \times \left(\frac{3}{4}\right)^{-1} \times 2^0$ **n.** $\left(\frac{9}{100}\right)^{-\frac{3}{2}} \times \left(\frac{4}{25}\right)^{\frac{3}{2}}$ **o.** $(27)^{1\frac{1}{3}} \times (81)^{-1\frac{1}{4}}$

p. $\left(11\frac{1}{9}\right)^{-\frac{1}{2}} \times \left(6\frac{1}{4}\right)^{-\frac{3}{2}}$ **q.** $(0.125)^{-\frac{1}{3}} \times (0.25)^{\frac{3}{2}}$ **r.** $(\sqrt[3]{8})^2 \times \sqrt[4]{16^3}$

4. Simplify and express using positive indices.

a. $\sqrt[3]{a^{-1}} \div \sqrt[3]{a}$ **b.** $\sqrt[5]{a^{-3}} \div \sqrt[5]{a^7}$ **c.** $\sqrt[3]{a^2} \div \sqrt[3]{a^{-3}}$

d. $\left(\sqrt[3]{x^5}\right)^{\frac{1}{2}} \times \sqrt[6]{x^{-5}}$ **e.** $\left\{\left(\sqrt{a^3}\right)^{-2}\right\}^{\frac{-1}{2}}$ **f.** $\left(\sqrt{x^2y^2}\right)^{-6}$

g. $\sqrt{\frac{4a^{-2}}{9x^2}}$ **h.** $\left(\sqrt[3]{27x^3}\right)^{-2}$ **i.** $\left(\frac{xy^{-1}}{\sqrt{x^5}}\right)^{-2}$

2.2 Solving Equations with Indices

$2^x = 2^3$ is an equation. Because the bases of the powers on either side of the equal sign are equal, the exponents must be equal. Thus, from $2^x = 2^3$, we can conclude that $x = 3$.

Similarly, on either side of the equation $x^5 = 2^5$ are powers with equal exponents.

Because the indices are equal, the bases are also equal. Therefore, from $x^5 = 2^5$ we can conclude that $x = 2$. If $x^2 = 3^2$ then the indices are equal but in this case, both $+\,3$ and $-\,3$ are solutions. The reason for two solutions arising is because the exponent " 2 " is an even number. In this lesson, we will only consider powers with

a positive base. Thus, in expressions of the form x^m, $x > 0$.
There is a special property of powers of 1. All powers of 1 are equal to 1. That is, for any m, $1^m = 1$.

Let us summarise the above observations.

For $x > 0$, $y > 0$, $y \neq 1$ and $x \neq 1$,

if $x \neq 0$ $x^m = x^n$, then $m = n$.
if $m \neq 0$ and $x^m = y^m$, then $x = y$.

Let us use these rules to solve equations with indices.

Example 1

Solve.

(i) $4^x = 64$ (ii) $x^3 = 343$ (iii) $3 \times 9^{2x-1} = 27^{-x}$

(i) $4^x = 64$

$4^x = 4^3$

$\therefore \underline{\underline{x = 3}}$

(ii) $x^3 = 343$

$x^3 = 7^3$

$\therefore \underline{\underline{x = 7}}$

(iii) $3 \times 9^{2x-1} = 27^{-x}$

$3 \times (3^2)^{2x-1} = (3)^{3(-x)}$

$3 \times 3^{2(2x-1)} = 3^{-3x}$

$3^{1+4x-2} = 3^{-3x}$

$\therefore 1 + 4x - 2 = -3x$

$4x + 3x = 2 - 1$

$7x = 1$

$\underline{\underline{x = \frac{1}{7}}}$

Exercise 2.2

1. Solve each of the following equations.

a. $3^x = 9$ **b.** $3^{x+2} = 243$

c. $4^{3x} = 32$ **d.** $2^{5x-2} = 8^x$

e. $8^{x-1} = 4^x$ **f.** $x^3 = 216$

g. $2\sqrt{x} = 6$ **h.** $\sqrt[3]{2x^2} = 2$

2. Solve each of the following equations.

a. $2^x \times 8^x = 256$ **b.** $8 \times 2^{x-1} = 4^{x-2}$

c. $5 \times 25^{2x-1} = 125$ **d.** $3^{2x} \times 9^{3x-2} = 27^{-3x}$

e. $4^x = \frac{1}{64}$ **f.** $(3^x)^{-\frac{1}{2}} = \frac{1}{27}$

g. $3^{4x} \times \frac{1}{9} = 9^x$ **h.** $x^2 = \left(\frac{1}{8}\right)^{-\frac{2}{3}}$

2.3 Laws of logarithms

We know that, using the laws of logarithms, we can write

$\log_2(16 \times 32) = \log_2 16 + \log_2 32$ and $\log_2(32 \div 16) = \log_2 32 - \log_2 16$. These laws, in general, can be written as follows.

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Let us learn another law of a similar type.

Consider $\log_5 125^4$ as an example.

$$\begin{aligned}\log_5 125^4 &= \log_5(125 \times 125 \times 125 \times 125)\\ &= \log_5 125 + \log_5 125 + \log_5 125 + \log_5 125\\ &= 4\log_5 125\end{aligned}$$

Similarly,

$$\log_{10} 10^5 = 5\log_{10} 10 \text{ and}$$
$$\log_3 5^2 = 2\log_3 5$$

This observation can be written, in general, as the following logarithmic law.

$$\boxed{\log_a m^r = r\log_a m}$$

This law is even valid for expressions with fractional indices. Given below are a few examples, where this law is applied to powers with fractional indices.

$$\log_2 3^{\frac{1}{2}} = \frac{1}{2}\log_2 3$$

$$\log_5 7^{\frac{2}{3}} = \frac{2}{3}\log_5 7$$

The following examples consider how all the laws of logarithms that you have learned so far, including the above, are used.

Example 1

Evaluate.

(i) lg1000 (ii) $\log_4 \sqrt[3]{64}$ (iii) $2\log_2 2 + 3\log_2 4 - 2\log_2 8$

(i) $\begin{aligned}\lg 1000 &= \lg 10^3\\ &= 3\lg 10\\ &= 3 \times 1 \quad \text{(because } \lg 10 = 1)\\ &= \underline{\underline{3}}\end{aligned}$

(ii) $\log_4 \sqrt[3]{64} = \log_4 64^{\frac{1}{3}}$

$= \frac{1}{3} \log_4 64$

$= \frac{1}{3} \log_4 4^3$

$= \frac{1}{3} \times 3 \log_4 4$

$= \log_4 4$

$= \underline{\underline{1}}$

(iii) $2 \log_2 2 + 3 \log_2 4 - 2\log_2 8 = 2\log_2 2 + 3 \log_2 2^2 - 2\log_2 2^3$

$= \log_2 2^2 + \log_2 (2^2)^3 - \log_2 (2^3)^2$

$= \log_2 \left(\frac{2^2 \times (2^2)^3}{(2^3)^2}\right)$

$= \log_2 \left(\frac{2^2 \times 2^6}{2^6}\right)$

$= \log_2 2^2$

$= 2 \log_2 2$

$= \underline{\underline{2}}$

Example 2

Solve.

(i) $2\lg 8 + 2\lg 5 = \lg 4^3 + \lg x$

$\therefore \lg x = 2\lg 8 + 2\lg 5 - \lg 4^3$

$= \lg 8^2 + \lg 5^2 - \lg 4^3$

$\therefore \lg x = \lg \left(\frac{8^2 \times 5^2}{4^3}\right)$

$\therefore \lg x = \lg 25$

$\therefore \underline{\underline{x = 25}}$

(ii) $2\log_b 3 + 3\log_b 2 - \log_b 72 = \frac{1}{2}\log_b x$

$\therefore\ 2\log_b 3 + 3\log_b 2 - \log_b 72 = \frac{1}{2}\log_b x$

$\therefore\ \log_b 3^2 + \log_b 2^3 - \log_b 72 = \log_b x^{\frac{1}{2}}$

$\therefore\ \log_b\left(\frac{3^2 \times 2^3}{72}\right) = \log_b x^{\frac{1}{2}}$

$\therefore\ \frac{3^2 \times 2^3}{72} = x^{\frac{1}{2}}$

$\therefore\ 1^2 = (x^{\frac{1}{2}})^2$

$\therefore\ 1 = x^1$

$\therefore\ \underline{\underline{x = 1}}$

Example 3

Verify: $\log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$

$$\begin{aligned}
\text{Left Side} &= \log_5 \frac{75}{3} \\
&= \log_5 25 \\
&= \log_5 5^2 \\
&= 2 \\
\text{Right Side} &= \log_5 40 - \log_5 8 + 1 \\
&= \log_5 \frac{40}{8} + 1 \\
&= \log_5 5 + 1 \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

$\therefore\ \log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$

Use the laws of logarithms to do the following exercise.

Exercise 2.3

1. Evaluate.

a. $\log_2 32$ **b.** $\lg 10000$ **c.** $\frac{1}{3}\log_3 27$

d. $\frac{1}{2}\log_5 \sqrt{25}$ **e.** $\log_3 \sqrt[4]{81}$ **f.** $3\log_2 \sqrt[3]{8}$

2. Simplify each of the following expressions and find the value.

a. $2\log_2 16 - \log_2 8$

b. $\lg 80 - 3\lg 2$

c. $2\lg 5 + 3\lg 2 - \lg 2$

d. $\lg 75 - \lg 3 + \lg 28 - \lg 7$

e. $\lg 18 - 3\lg 3 + \frac{1}{2}\lg 9 + \lg 5$

f. $4\lg 2 + \lg\frac{15}{4} - \lg 6$

g. $\lg\frac{1}{256} - \lg\frac{125}{4} - 3\lg\frac{1}{20}$

h. $\log_3 27 + 2\log_3 3 - \log_3 3$

i. $\lg\frac{12}{5} + \lg\frac{25}{21} - \lg\frac{2}{7}$

j. $\lg\frac{3}{4} - 2\lg\frac{3}{10} + \lg 12 - 2$

3. Solve the following equations.

a. $\log x + \lg 4 = \lg 8 + \lg 2$

b. $4\lg 2 + 2\lg x + \lg 5 = \lg 15 + \lg 12$

c. $3\lg x + \log 96 = 2\lg 9 + \lg 4$

d. $\lg x = \frac{1}{2}(\lg 25 + \lg 8 - \lg 2)$

e. $3\lg x + 2\lg 8 = \lg 48 + \frac{1}{2}\lg 25 - \lg 30$

f. $\lg 125 + 2\lg 3 = 2\lg x + \lg 5$

Summary

- $\sqrt[n]{a} = a^{\frac{1}{n}}$
- If $x > 0, y > 0$ and $x \neq 1, y \neq 1$

  $x \neq 0$ and $x^m = x^n$, then $m = n$.
  $m \neq 0$ and $x^m = y^m$, then $x = y$.
- $\log_a m^r = r\log_a m$

Miscellaneous Exercise

1. Find the value.

a. $(\sqrt[3]{8})^2 \times \frac{1}{\sqrt[3]{27}}$

b. $(\sqrt{8})^3 \times \frac{1}{\sqrt[3]{27}} \times 6^{-\frac{5}{2}}$

c. $\dfrac{32^{-\frac{2}{5}} \times 216^{\frac{2}{3}}}{81^{\frac{3}{4}} \times \sqrt[3]{8^0} \times \sqrt[3]{27^{-2}}}$

d. $\sqrt{\dfrac{18 \times 5^2}{8}}$

e. $\left(\dfrac{1}{8}\right)^{-\frac{1}{3}} \times 5^{-2} \times 100$

f. $27^{\frac{2}{3}} - 16^{\frac{3}{4}}$

2. Simplify and express using positive indices.

a. $\sqrt{a^2b^{-\frac{1}{2}}}$

b. $(x^{-4})^{\frac{1}{2}} \times \dfrac{1}{\sqrt{x^{-3}}}$

c. $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})\ (x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d. $(x \div \sqrt[n]{x})^n$

e. $\left[\left(\sqrt{a^3}\right)^{-2}\right]^{\frac{1}{2}}$

3. Verify the following.

a. $\lg\left(\dfrac{217}{38} \div \dfrac{31}{266}\right) = 2 \lg 7$

b. $\dfrac{1}{2} \lg 9 + \lg 2 = 2 \lg 3 - \lg 1.5$

c. $\log_3 24 + \log_3 5 - \log_3 40 = 1$

d. $\lg 26 + \lg 119 - \lg 51 - \lg 91 = \lg 2 - \lg 3$

e. $2 \log_a 3 + \log_a 20 - \log_a 36 = \log_a 10 - \log_a 2$