1 Real Numbers

By studying this lesson will be able to

- to investigate number sets
- workout basic mathematical operations regarding surds.

It is believed that the concept of numbers originated among the human race about 30 000 years ago. This concept which originated and developed independently in various civilizations, evolved globally and has now become a universal field of study named mathematics.

It can be assumed that numbers were initially used in early civilizations for simple purposes such as counting and accounting. There is no doubt that the first numerical concepts that were developed were “one” and “two”. Later the concepts of three, four etc., must have been developed. Then man would have realized that he could name any amount that he wished in this manner. Different civilizations used different symbols to name numbers.

It is accepted based on historical evidence, that the numerals 1, 2, 3 etc., which we now use, originated in India. The honour of being the first to use the concept of zero as a number as well as being the first to introduce the positional decimal number system also goes to India. This number system is now defined as the Hindu-Arabic number system and the modern belief is that it was first taken to the Middle-East and then to Europe by traders. This system is the standard number system which is accepted and used worldwide now.

The manipulation of numbers using the basic mathematical operations (addition, subtraction, multiplication and division) can be considered as a great revolution in the history of mankind in relation to the use of numbers. In this age of technology it is unimaginable to think of the existence of man without numbers and the operations performed on them.

Although the numbers 1, 2, 3 etc., can be considered as the first numbers that were used to fulfill certain needs of man, later the number zero, fractional numbers and negative numbers were also included in the number system. During the period when mathematics was developing as a separate field, the attention of mathematicians was directed towards various other types of numbers (sets) too. In this lesson we hope to study about such sets of numbers, their notations and properties.

The Set of Integers ($\mathbb{Z}$)

It is natural that we identify initially the numbers 1, 2, 3, … which we first learnt about as children. These numbers are defined as **counting numbers** and the set which consists of all these numbers is written using set notation as follows.

$$\{1, 2, 3, \ldots\}$$

The reason for this set of numbers to be called the counting numbers is very clear. However its mathematical usage in modern times is limited. The name used most often now for this set is "**the set of positive integers**". This set is denoted by $\mathbb{Z}^+$.

$$\text{Thus, } \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

That is, the numbers 1, 2, 3, … are called positive integers.

The numbers defined as negative integers are -1, -2, -3, etc. Although there is no commonly used symbol to denote this set, some mathematicians, based on the needs of their field of study, use the symbol $\mathbb{Z}^-$.

The positive integers, zero and the negative integers together form the set of **integers**. This set is denoted by $\mathbb{Z}$. Accordingly,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

or equivalently,

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

The Set of Natural Numbers ( )

Let us consider again the set of numbers 1, 2, 3,... This set is also defined as the set of **natural numbers** and is denoted by .

That is,   = {1, 2, ...}.

Note: There is no consensus among mathematicians regarding which numbers should be considered as natural numbers. The suitability of calling the numbers 1, 2, 3, … natural numbers is clear. However some of the mathematicians (especially specialists in set theory), have considered 0 as a natural number in their books.

One reason may be because at that time there was no accepted name nor accepted symbol for the set consisting of 0 and the positive integers.However most books on number theory consider the set of natural numbers to be the set $\{1, 2, 3, ...\}$.Almost all authors of mathematics books now mention at the beginning of their books which set of numbers they consider as the natural numbers.

The set of Rational Numbers $(\mathbb{Q})$

We have come across earlier that, like the integers, fractions too can be considered as numbers, and that operations such as addition and multiplication can be performed on them too. Every integer can be written as a fraction. (For example, we can write $2 = \frac{2}{1}$). Further, a fraction can be written in different forms, all having the same numerical value. (For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$). We have also come across negative fractions $(-\frac{2}{5}, -\frac{11}{3}$, etc.). Although we usually think that the numerator and denominator of a fraction should consist of integers, this is actually not the case. For example, $\frac{3}{\sqrt{2}}$ is also a fraction. However, fractions with integers in both the numerator and the denominator (apart from 0 in the denominator), have an important place in mathematics. They are called **rational numbers**. The set of rational numbers is denoted by $\mathbb{Q}$. Accordingly, the set of rational numbers can be defined using the set builder notation as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \;:\; a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

There are other ways too of defining the set of rational numbers. One other way is as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \;:\; a \in \mathbb{Z}, b \in \mathbb{Z}^{+} \right\}.$$

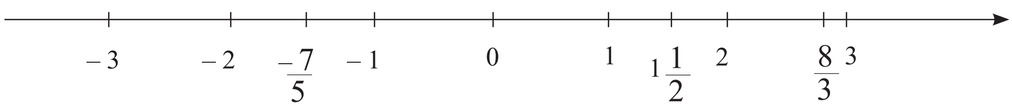
Both these definitions are equivalent. Since the denominator of a rational number cannot be 0 and since all the negative rational numbers can be obtained by considering the fractions with the negative integers in the numerator and positive integers in the denominator.

The Set of Irrational Numbers (Q')

It is appropriate to define the irrational numbers now. Do you recall how you learnt about numbers in previous grades by drawing a number line? Let us reconsider this now.

Let us consider a straight line which can be lengthened as required in either direction. Let us name a point we like on that line as the origin 0. Let us assume that we have marked all the numbers 1, 2, 3, etc., on one side of 0 (usually the right hand side) and all the numbers – 1, – 2, – 3 etc., on the opposite side, keeping equal gaps between the numbers. That is, let us assume that the points corresponding to all the integers have been marked on this number line. Let us also assume that the points corresponding to all the rational numbers too have been marked on this line.

The figure below shows several such points that have been marked.



Accordingly, all the rational numbers (including the integers) are now assumed to have been marked on this line. Now, do you think that corresponding to each point on the line, a number has been marked? If asked differently, do you think that the distance from 0 to each point on the line can be written as a rational number? In truth, there are several points remaining on the number line which have not been marked. That is, there are points remaining on this number line that cannot be represented by a rational number. It is clear that the points that are remaining are those which correspond to the numbers which **cannot** be written in the form $\frac{a}{b}$, where a and b are integers. The numbers which correspond to the remaining points are defined as irrational numbers.

There is no specific symbol to denote the set of irrational numbers and it is usually denoted by Q', the complement of the set Q. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ can be given as examples of irrational numbers. In fact, the square root of any positive integer which is not a perfect square is an irrational number. Apart from these, mathematicians have proved that π, which is the ratio of the circumference of any circle to its diameter, is also an irrational number. We take the value of π to be $\frac{22}{7}$ as an approximate value, for the convenience of performing calculations.

The Set of Real Numbers ($\mathbb{R}$)

According to the above discussion, all the numbers on a number line can be represented by rational numbers or irrational numbers. We call all the rational numbers together with all the irrational numbers, that is, all the numbers that can be represented on a number line, the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$.

The Decimal Representation of a Number

Any real number can be represented as a decimal number. Initially, let us consider the decimal representation of several rational numbers.

1. The decimal representation of a rational number

$$\frac{1}{2} = 0.5 = 0.5000\ ...$$

$$4 = 4.000\ ...$$

$$\frac{11}{8} = 1.375 = 1.375000\ ...$$

$$\frac{211}{99} = 2.131313...$$

$$\frac{767}{150} = 5.11333...$$

$$\frac{37}{7} = 5.285714285714285714\ ...$$

A common property of these decimal representations is that starting at a certain point to the right of the decimal point (or from the beginning), one set of numerals (or one numeral) is recurring. Recurring means that it keeps repeating itself.

For example, in the decimal representation of $\frac{1}{2}$, the numeral 0 recurs starting from the second decimal place. The numeral 0 recurs from the first decimal place in the decimal representation of 4, the pair of numerals 13 recurs from the beginning in the decimal representation of $\frac{211}{99}$ and the group of numerals 285714 recurs from the beginning in the decimal representation of $\frac{37}{7}$. This property, that is, a group of numerals recurring continuously, is a property common to all rational numbers. If the portion that recurs is just 0, such a decimal representation is defined as a **finite decimal** (or **terminating decimal**). The decimals of which the portion that recurs is not zero, are called **recurring decimals**. Accordingly, $\frac{1}{2}$, 4 and $\frac{11}{8}$ in the above example are finite decimals while the rest are recurring decimals.

The above discussion leads to the following statement.

Every rational number can be written as a finite decimal or a recurring decimal.

Let us now learn a marvelous result regarding rational numbers. Suppose the rational number $\frac{a}{b}$ has a finite decimal representation. Let us assume that a and b have no common factors. Then the denominator (that is, b) has only powers of 2 or 5 (or both) as its factors. A rational number which has a recurring decimal representation must have a prime factor other than 2 and 5 in its denominator.

Recurring decimals are written in a concise form, by placing a dot above a numeral or numerals as shown in the following examples to indicate that they are recurring.

Recurring Decimal	Written Concisely
12.4444	$12.\dot{4}$
2.131313...	$2.\dot{1}\dot{3}$
5.11333...	$5.11\dot{3}$
5.285714285714285714...	$5.\dot{2}8571\dot{4}$

Exercise 1.1

1. For each of the following rational numbers state whether it is a finite decimal or a recurring decimal. Express the fractions which are recurring decimals in decimal form and then write them in a concise form.

a. $\frac{3}{4}$ **b.** $\frac{5}{5}$ **c.** $\frac{3}{7}$ **d.** $\frac{5}{9}$ **e.** $\frac{5}{21}$ **f.** $\frac{7}{32}$

g. $\frac{19}{33}$ **h.** $\frac{13}{50}$ **i.** $\frac{7}{64}$ **j.** $\frac{5}{18}$ **k.** $\frac{15}{128}$ **l.** $\frac{41}{360}$

2. The decimal representation of an irrational number

Finally, let us consider the decimal representation of an irrational number. In the decimal representation of an irrational number, no group of numerals recurs. For example, when the decimal representation of $\sqrt{2}$ is written down up to 60 decimal places, we obtain the following:

1.414213562373095048801688724209698078569671875376948073176679

π, which is a number we come across often is also an irrational number. When the value of π is calculated up to 60 decimal places we obtain the following:

3.141592653589793238462643383279502884197169399375105820974944

The following statements can be made regarding irrational numbers.

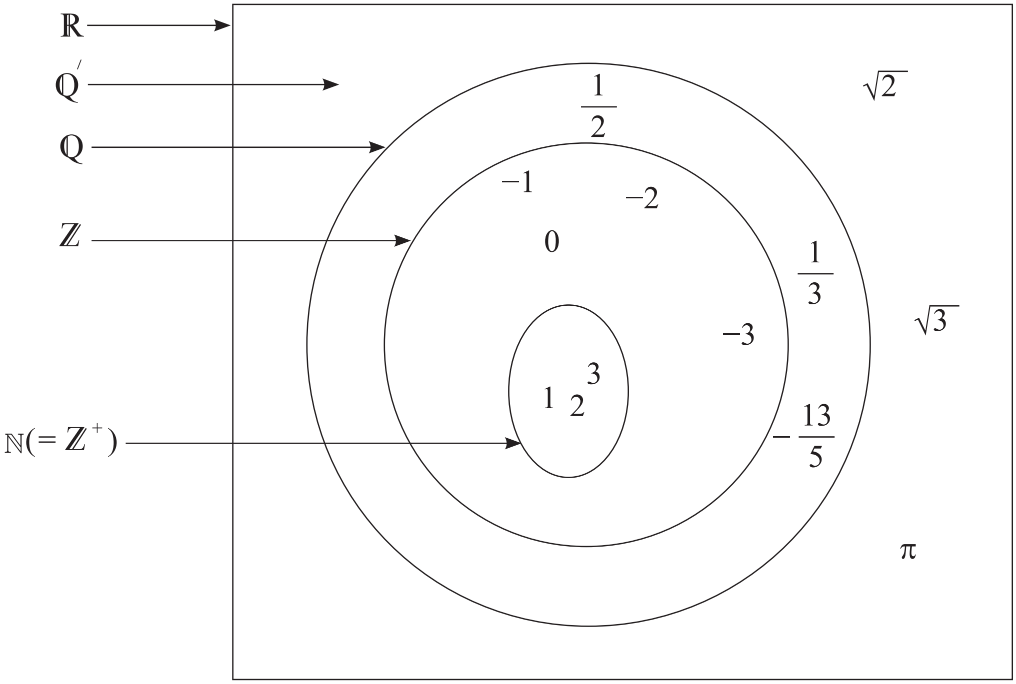
There is no group of recurring numerals in the decimal representation of an irrational number. If the decimal representation of a number is not finite, it is called an infinite decimal. Accordingly rational numbers which are recurring decimals and irrational numbers have infinite decimal representations.

Note: When describing the decimal representation of irrational numbers, one frequent error that is made is stating that "there is no pattern in the decimal representation of an irrational number". The issue here is that the word "pattern" is not well defined in mathematics. For example, the following decimal number has a very clear pattern.

0.101001000100001000001...

However, this is an irrational number, Observe that there is no group of recurring numbers.

All the sets of numbers that have been studied so far can be represented in a Venn diagram as follows, with the set of real numbers as the universal set, and the other sets as its subsets. A few numbers belonging to each subset have been included to make it easier for you to understand the relationship between the sets.



Exercise 1.2

1. State whether the following real numbers as rational numbers or irrational numbers.

a. $\sqrt{2}$ **b.** $\sqrt{25}$ **c.** $\sqrt{6}$ **d.** $\sqrt{11}$ **e.** 6.52

2. Determine whether each of the following statement is true or false.

a. Any real number is a finite decimal or an infinite decimal.
b. There can be rational numbers with infinite decimals representations.
c. Any real number is a recurring decimal or an infinite decimal.
d. 0.010110111011110... is a rational number.

1.2 Surds

There is no doubt that you recall how numerical (and algebraic) expressions are written using the symbol " $\sqrt{}$ " which is defined as the radical sign. For example, $\sqrt{4}$ is defined as the positive square root of 4 and it represents the positive number which when squared is equal to 4; that is 2. The positive square root is referred to as square root too. If a certain positive integer x is such that its square root $\sqrt{x}$ is also a positive integer, then x is defined as a perfect square. Accordingly, 4 is a perfect square. Since $\sqrt{4}$ is equal to 2. However, $\sqrt{2}$ is not the square root of a perfect square. We observed earlier that $\sqrt{2}$ is approximately equal to 1.414. We also learnt earlier in this lesson that $\sqrt{2}$ is an irrational number. A numerical term involving the symbol $\sqrt{}$ of which the value is not an integer is defined as a surd. The radical sign "$\sqrt{}$ " is used not only to denote the square roots of numbers, but also other roots. For example, $\sqrt[3]{2}$ denotes the positive number which when raised to the power 3 is equal to 2. This is called the cube root of 2. This is also an irrational number. Its value is approximately 1.2599. (You can verify this by finding the value of 1.2599^3). We can define the fourth root of 2 and the fifth root of 2 etc., in a similar manner. Such definitions can be made for other positive numbers as well; for example, $\sqrt[3]{5}$ and $\sqrt[6]{8.24}$. Such expressions are also surds. However, in this lesson we will only consider surds which are square roots of positive integers.

The square root of a positive integer which is not a perfect square is neither a finite decimal nor a recurring decimal. Observe that surds are therefore always irrational numbers.

Here our focus is on simplifying expressions which involve surds. There are many reasons why such simplifications are important. One reason is that it facilitates calculations. For example, when it is necessary to find the value of $\frac{1}{\sqrt{2}}$, if we take $\sqrt{2}$ to be approximately equal to 1.414, we would need to find the value of $\frac{1}{1.414}$. This division is fairly long. However, by simplifying this in the following manner, calculations are made easier.

$$\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \quad \text{(Multiplying the numerator and denominator by } \sqrt{2}\text{)}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1.414}{2} = 0.707.$$

Another reason for simplifying surds is to minimize errors during calculations. For example, let us find the value of $\frac{\sqrt{20}}{2} - \sqrt{5}$. Let us use 4.5 as an approximate value for $\sqrt{20}$, and 2.2 as an approximate value for $\sqrt{5}$. Then

$$\frac{\sqrt{20}}{2} - \sqrt{5} = \frac{4.5}{2} - 2.2$$
$$= 2.25 - 2.2$$
$$= 0.05$$

However, the actual value of this expression is 0. One reason for getting a different answer is because we used approximate values for $\sqrt{20}$ and $\sqrt{5}$. However, by simplifying the above expression in a different way, we can get the correct value which is 0.

Surds appear in various form.

A special feature of a surd of the form $\sqrt{20}$ is that the whole number is under the radical sign. Such surds are defined as **entire surds**. $6\sqrt{15}$ means $6 \times \sqrt{15}$ .This is a product of a surd and a rational number (not equal to 1). This is not an entire surd.
A surd is in its simplest form when it is in the form $a\sqrt{b}$; where a is a rational number and b has no factors which are perfect squares. For example $6\sqrt{15}$ is in its simplest form but $5\sqrt{12}$ is not, since 4 which is a perfect square is a factor of 12.

Now let us consider how to simplify expressions that contain surds of various forms.

Example 1

Simplify $3\sqrt{5} + 6\sqrt{5}$.

This can be simplified by considering $\sqrt{5}$ to be an unknown term.

$$3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}.$$

This simplification is similar to the simplification $3x + 6x = 9x$.

Observe that the expression obtained above in surd form cannot be simplified further. Keep in mind that simplifying further by using an approximate value for $\sqrt{5}$ is not what is meant by simplifying surds.
You should also keep in mind the important fact that an expression of the form $3\sqrt{2} + 8\sqrt{3}$ cannot be simplified further.

Now let us through examples, consider how expressions with surds are simplified by applying the properties of indices.

Example 2

Simplify the entire surd $\sqrt{20}$.

$$\begin{aligned}\sqrt{20} &= \sqrt{4\times 5} \\ &= \sqrt{4}\times\sqrt{5} \quad \left(\text{Since} \sqrt{ab}=\sqrt{a}\times\sqrt{b}\right) \\ &= 2\times\sqrt{5} \\ &= \underline{\underline{2\sqrt{5}}}\end{aligned}$$

Example 3

Express the surd $4\sqrt{5}$ as an entire surd.

$$\begin{aligned}4\sqrt{5} &= \sqrt{16}\times\sqrt{5} \quad (\text{Since } 4=\sqrt{16}) \\ &= \sqrt{16\times 5} \\ &= \underline{\underline{\sqrt{80}}}\end{aligned}$$

Next let us consider how multiplication and division are performed on surds.

Example 4

Simplify: $5\sqrt{3}\times 4\sqrt{2}$.

Let us multiply the rational and irrational parts separately.

$$\begin{aligned}5\sqrt{3}\times 4\sqrt{2} &= 5\times 4\times\sqrt{3}\times\sqrt{2} \\ &= 20\times\sqrt{3\times 2} \\ &= \underline{\underline{20\sqrt{6}}}\end{aligned}$$

Example 5

Simplify: $3\sqrt{20}\div 2\sqrt{5}$.

The surd $3\sqrt{20}$ can be written as $3\sqrt{4\times 5}$. Simplifying further, it can be written as $3\times 2\sqrt{5} = 6\sqrt{5}$

$$\therefore\ 3\sqrt{20}\div 2\sqrt{5} = \frac{3\sqrt{20}}{2\sqrt{5}} = \frac{6\sqrt{5}}{2\sqrt{5}}$$

$$= \underline{\underline{3}}$$

Next we will consider how expressions of the form $\frac{a}{\sqrt{b}}$ are simplified. Examples for such expressions are $\frac{3}{\sqrt{2}}$ and $\frac{4}{\sqrt{5}}$. Expression of this form has a square root term in the denominator.

Now let us consider how such an expression can be converted into an expression with an integer (or a rational number) in the denominator.

Example 6

Express $\frac{3}{\sqrt{2}}$ as a fraction with an integer in the denominator.

The method used here is, to multiply both the numerator and the denominator of $\frac{3}{\sqrt{2}}$ by $\sqrt{2}$.

$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \underline{\underline{\frac{3\sqrt{2}}{2}}}$$

This process is defined as **rationalizing** the denominator.

Example 7

Rationalise the denominator of $\frac{a}{\sqrt{b}}$

$$\frac{a}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}}$$

$$= \underline{\underline{\frac{a\sqrt{b}}{b}}}$$

Now let us consider how an expression involving surds is simplified.

Example 8

Simplify $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28}$.

$$4\sqrt{63} = 4 \times \sqrt{9 \times 7} = 4 \times 3\sqrt{7}$$

$$= 12\sqrt{7}$$

$$8\sqrt{28} = 8 \times \sqrt{4 \times 7} = 8 \times 2\sqrt{7}$$

$$= 16\sqrt{7}$$

Therefore $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28} = 12\sqrt{7} - 5\sqrt{7} - 16\sqrt{7}$

$$= \underline{\underline{-9\sqrt{7}}}$$

Let us consider how a more complex expression involving surds is simplified.

Example 9

Simplify $\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}}$

$$\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}} = \frac{2\sqrt{2\times 3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4\times 3}}$$

$$= \frac{2\sqrt{2}\times\sqrt{3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4\times 3}}$$

$$= 2\sqrt{3} + 5\sqrt{3} - \frac{3}{2\sqrt{3}}$$

$$= 7\sqrt{3} - \frac{3\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}}$$

$$= 7\sqrt{3} - \frac{3\sqrt{3}}{2\times 3}$$

$$= 7\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{13\sqrt{3}}{2}$$

Exercise 1.3

1. Convert the following entire surds into surds.

a. $\sqrt{20}$ **b.** $\sqrt{48}$ **c.** $\sqrt{72}$ **d.** $\sqrt{28}$

e. $\sqrt{80}$ **f.** $\sqrt{45}$ **g.** $\sqrt{75}$ **h.** $\sqrt{147}$

2. Convert the following surds into entire surds.

a. $2\sqrt{3}$ **b.** $2\sqrt{5}$ **c.** $4\sqrt{7}$ **d.** $5\sqrt{2}$ **e.** $6\sqrt{11}$

3. Simplify.

a. $\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$

b. $\sqrt{5} + 2\sqrt{7} + 2\sqrt{5} - 3\sqrt{7}$

c. $4\sqrt{3} + 5\sqrt{2} + 3\sqrt{5} - 3\sqrt{2} + 3\sqrt{5} - 2\sqrt{3}$

d. $6\sqrt{11} + 3\sqrt{7} - 2\sqrt{11} - 5\sqrt{7} + 4\sqrt{7}$

e. $8\sqrt{3} + 7\sqrt{7} - 2\sqrt{3} + 3\sqrt{7} - 3\sqrt{7}$

4. Rationalise the denominator of the following fractions.

a. $\frac{2}{\sqrt{5}}$ b. $\frac{5}{\sqrt{3}}$ c. $\frac{5}{\sqrt{7}}$ d. $\frac{12}{2\sqrt{3}}$ e. $\frac{27}{3\sqrt{2}}$

f. $\frac{3}{2\sqrt{5}}$ g. $\frac{3\sqrt{5}}{2\sqrt{7}}$ h. $\frac{2\sqrt{3}}{3\sqrt{2}}$ i. $\frac{3\sqrt{3}}{2\sqrt{5}}$

5. Simplify.

a. $3\sqrt{2} \times 2\sqrt{3}$ b. $5\sqrt{11} \times 3\sqrt{7}$ c. $\sqrt{5} \times 3\sqrt{3}$

d. $4\sqrt{7} \div 2\sqrt{14}$ e. $6\sqrt{27} \div 3\sqrt{3}$ f. $\sqrt{48} \div 5\sqrt{3}$

6. Simplify.

a. $2\sqrt{27} - 3\sqrt{3} + 4\sqrt{7} + 3\sqrt{28}$

b. $3\sqrt{63} - 2\sqrt{7} + 3\sqrt{27} + 3\sqrt{3}$

c. $2\sqrt{128} - 3\sqrt{50} + 2\sqrt{162} + \frac{4}{\sqrt{2}}$

d. $\sqrt{99} - 2\sqrt{44} + \frac{110}{\sqrt{11}}$

e. $\frac{\sqrt{20}}{2} - \sqrt{5}$