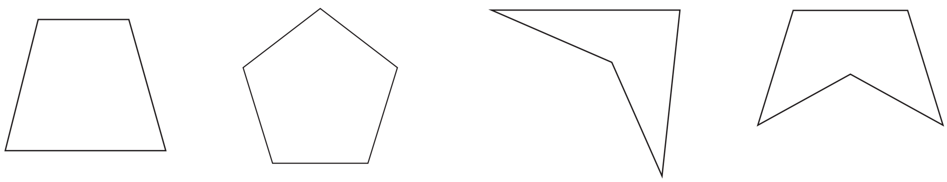
25 Angles of a Polygon

By studying this lesson you will be able to;

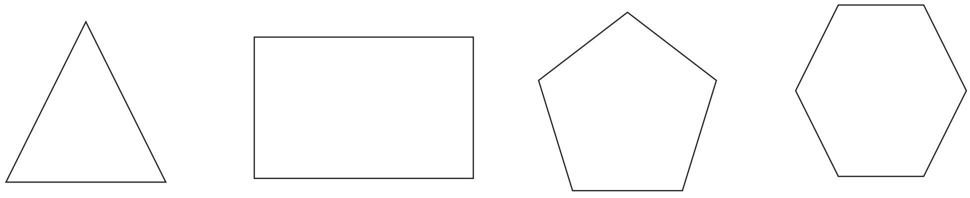
- solve geometrical problems related to the interior angles of a polygon,
- solve geometrical problems related to the exterior angles of a polygon,
- solve problems related to regular polygons.

A plane figure bounded by three or more straight line segments is known as a polygon. The two main types of polygons are convex polygons and concave polygons.



Convex Polygons Concave Polygons

Some polygons have special names by which they are identified, which depend on the number of sides they have. Accordingly, polygons which have 3, 4, 5 and 6 sides respectively are known as triangles, quadrilaterals, pentagons and hexagons.



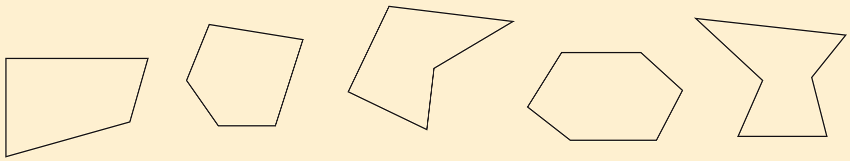
In previous grades you have learnt the following results on the sum of the interior angles of a triangle and of a quadrilateral.

The sum of the interior angles of a triangle is 180^{o}.
The sum of the interior angles of a quadrilateral is 360^{o}.

Do the following review exercise to further establish the above facts learnt on polygons in previous grades.

Review Exercise

1. From the given figures, select all the convex polygons.



(i) (ii) (iii) (iv) (v)

2. From the statements given below, select the true statements.

a. A polygon with 7 sides is known as a heptagon.

b. In any polygon, the number of interior angles is equal to the number of sides.

c. A polygon with all sides equal is known as a regular polygon.

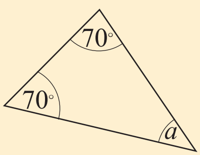
d. At each vertex of a polygon, the sum of the interior angle and the exterior angle is 180°.

e. There are 11 interior angles in a decagon.

f. The sum of the exterior angles of a quadrilateral is 180°.

3. Find the magnitude of each of the angles denoted by a lower case letter in each of the following figures.

i.



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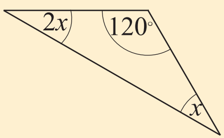

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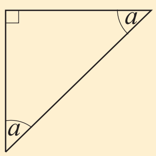

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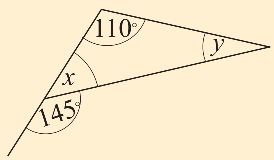

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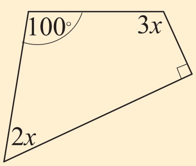

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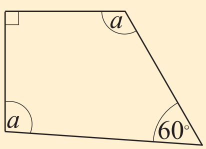
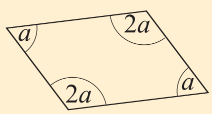
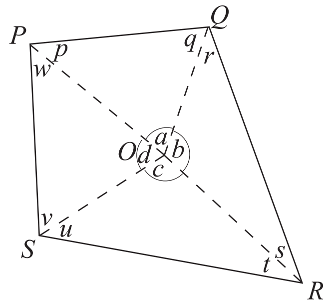


vii.



25.1 The sum of the interior angles of a polygon

Let us first consider how to find the sum of the interior angles of a quadrilateral.



In the given figure, O is any point within the quadrilateral $PQRS$. By joining PO, QO, RO and SO we obtain 4 triangles.

Since the sum of the interior angles of a triangle is 180°,

considering the triangle PQO we obtain, $p + q + a = 180^{\circ}$,

considering the triangle QRO we obtain, $r + s + b = 180^{\circ}$,

considering the triangle RSO we obtain, $t + u + c = 180^{\circ}$,

considering the triangle SPO we obtain, $v + w + d = 180^{\circ}$,

By adding these 4 equations we obtain,

$$(p + q + a) + (r + s + b) + (t + u + c) + (v + w + d) = 180^{\circ} \times 4$$

$$\therefore\ (p + q + r + s + t + u + v + w) + (a + b + c + d) = 720^{\circ}$$

Since a, b, c and d are angles around the point O, $a + b + c + d = 360^{\circ}$.

$$\therefore\ p + q + r + s + t + u + v + w = 720^{\circ} - 360^{\circ}$$

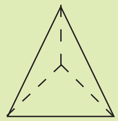
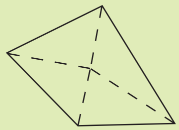
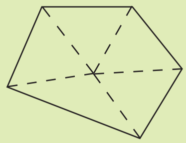
$$= 360^{\circ}$$

$\therefore$ the sum of the interior angles of a quadrilateral is 360°.

Now, to obtain an expression in terms of n for the sum of the interior angles of a polygon which has n sides, let us engage in the following activity.

Activity 1

Copy the following table and complete it.

Polygon	Figure	Number of triangles	Sum of the interior angles
Triangle		3	$180^{\circ} \times 3 - 360^{\circ} = 180^{\circ}$
Quadrilateral		4	$180^{\circ} \times 4 - 360^{\circ} = 360^{\circ}$
Pentagon		5	$180^{\circ} \times \ldots - 360^{\circ} = 540^{\circ}$
Hexagon		…………	…………………………………
Heptagon		…………	…………………………………
Octagon		…………	…………………………………
Polygon with n sides		…………	…………………………………

In the above activity, you must have obtained that when the number of sides is n, the sum of the interior angles of the polygon is $180^{\circ} \times n - 360^{\circ}$.

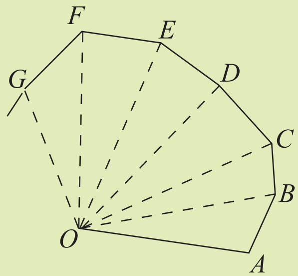
Let us write the expression $180^{\circ} \times n - 360^{\circ}$ as follows so that it is easier to remember.

$$
\begin{aligned}
180^{\circ} \times n - 360^{\circ} &= 90^{\circ} \times 2n - 90^{\circ} \times 4 \\
&= 90^{\circ}\,(2n - 4) \\
&= (2n - 4) \text{ right angles.}
\end{aligned}
$$

$\therefore$ The sum of the interior angles of a polygon of n sides $= (2n - 4)$ right angles.

Activity 2

2. Let us find a formula for the sum of the interior angles of a polygon in another way.



Polygon	Number of sides	Name of the polygon	Number of triangles	Sum of the interior angles
OAB	3	Triangle	1	$180^{o} \times 1 \ = 180^{o}$
OABC	4	Quadrilateral	2	$180^{o} \times$ .... = ...
OABCD	....	....	....	....
OABCDE	....	....	....	....
OABCDEF	....	....	....	....
OABCDEFG	....	....	....	....

i. Consider the above table and find in terms of n, the number of triangles that are formed by joining one vertex of an n-sided polygon to the other vertices of the polygon.

ii. Show that the sum of the interior angles of a polygon of n sides is $180^{o}\,(n-2)$.

Note: Historically, mathematicians such as the Greek mathematician Euclid, expressed the magnitudes of angles in terms of right angles. For example, the magnitude of the angle on a straight line was said to be 2 right angles and the sum of the interior angles of a quadrilateral was said to be 4 right angles. Accordingly, we can say that the sum of the interior angles of a polygon of n sides is $2n-4$ right angles. However, since we use degrees to measure angles and are familiar with the fact that a right angle is 90^{o}, the sum of the interior angles of a polygon can be remembered as either $90^{o}(2n-4)$ or $180^{o}(n-2)$ or any other equivalent expression which is easy to recall.

Example 1

Find the sum of the interior angles of a nonagon.

Sum of the interior angles of a polygon of n sides $= 180^\circ (n-2)$

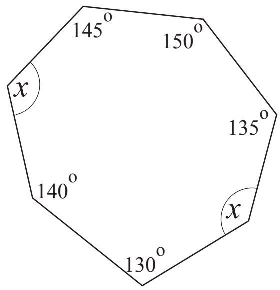
$\therefore$ the sum of the interior angles of a polygon of 9 sides $= 180^\circ (9-2)$

$= 180^\circ \times 7$

$= \underline{\underline{1260^\circ}}$

Example 2

Find the value of x based on the information in the figure.



Number of sides in the polygon $= 7$

$\therefore$ sum of the interior angles $= 180^\circ \ (7-2)$

$= 180^\circ \times 5$

$= 900^\circ$

$\therefore 145^\circ + 150^\circ + 135^\circ + x^\circ + 130^\circ + 140^\circ + x = 900^\circ$

$700^\circ + 2x = 900^\circ$

$2x = 900^\circ - 700^\circ$

$2x = 200^\circ$

$x = \dfrac{200^\circ}{2} = 100^\circ$

Example 3

The sum of the interior angles of a polygon is 1440°. Find the number of sides it has.

If the number of sides is n, the sum of the interior angles $= 180^\circ (n-2)$

$\therefore \ 180^\circ (n-2) = 1440^\circ$

$n - 2 = \dfrac{1440^\circ}{180} = 8$

$n - 2 = 8$

$n = 10$

$\therefore$ the number of sides $= 10$.

Exercise 25.1

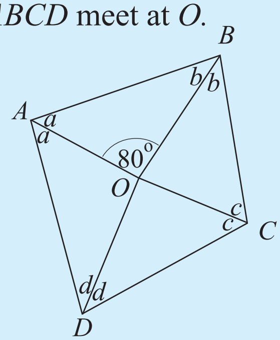
1. Find the sum of the interior angles of each of the polygons given below.

i. Pentagon **ii.** Octagon **iii.** Dodecagon **iv.** Polygon with 15 sides

2. Four of the interior angles of a heptagon are 100°, 112°, 130° and 150°. The remaining angles are equal. Find their magnitude.

3. The bisectors of the interior angles of the quadrilateral *ABCD* meet at *O*.

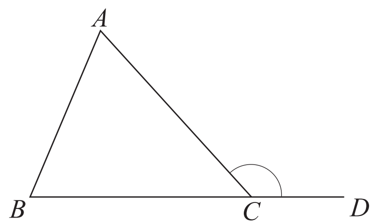
i. Find the value of $a + b + c + d$.
ii. Find the value of $a + b$.
iii. Find the value of $c + d$.
iv. Find the value of $C\hat{O}D$.



4. **i.** If the sum of the interior angles of a polygon is 1620°, find the number of sides it has.
ii. If the sum of the interior angles of a polygon is 3600°, find the number of sides it has.

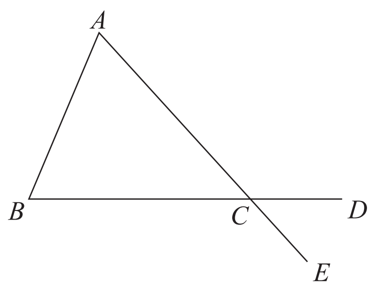
25.2 Sum of the exterior angles of a polygon

First, let us find the sum of the exterior angles of a triangle.



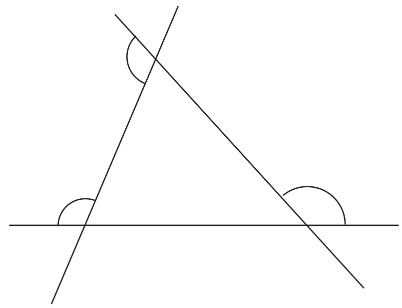
The side *BC* of the triangle *ABC* has been produced and point *D* has been marked on the produced line. The angle $A\hat{C}D$, with the straight line segment *CD* and the side *AC* as arms is an exterior angle of this triangle.

As indicated in the figure given below, by producing the side *AC* too we obtain an exterior angle.

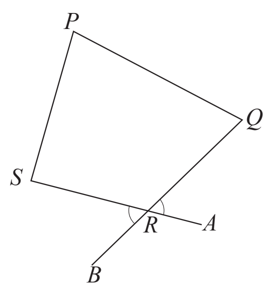


Since vertically opposite angles are equal, this exterior angle is equal in magnitude to the exterior angle $A\hat{C}D$. Either of these two angles can be considered as the exterior angle drawn at the vertex C of the triangle. However, $D\hat{C}E$ is not considered as an exterior angle.

As done above, exterior angles can be drawn at the vertices A and B of the triangle too.



We can define the exterior angles of a quadrilateral similarly.



By producing the side SR of the quadrilateral $PQRS$ up to A we obtain the exterior angle $Q\hat{R}A$ and by producing the side QR up to B we obtain the exterior angle $S\hat{R}B$. Since vertically opposite angles are equal, these two exterior angles are equal.

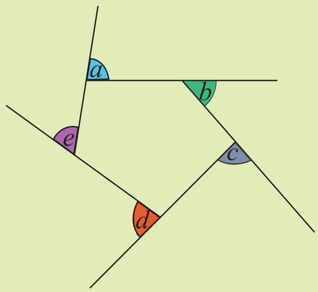
Furthermore, $A\hat{R}B$ is not considered as an exterior angle.

We can define the exterior angles of any polygon similarly.

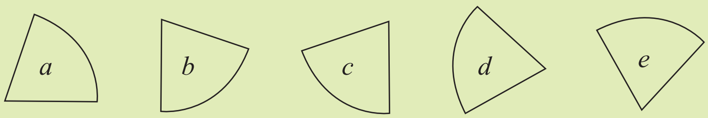
Now, through the following activity, let us determine a value for the sum of the exterior angles of a polygon.

Activity 3

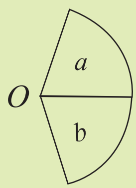
Step 1: Draw a pentagon on a half sheet and name its exterior angles.



Step 2: Using a blade, cut out the exterior angles as laminas (in the form of sectors of circles) and separate them. (By using the same radius when drawing the sectors, the final outcome will be neat)



Step 3: On a sheet of paper, paste the laminas which were cut out, such that their vertices meet at one point and such that they don't overlap each other.



Step 4: Carry out the above steps for a hexagon and a heptagon too.

Step 5: Write the common characteristic of the figures obtained by pasting the exterior angles of the polygons and write what can be concluded through this activity too.

You may have observed in the above activity that for each polygon, the exterior angles cover the angle around a point. Accordingly, it can be concluded that the sum of the exterior angles of a polygon is equal to the sum of the angles around a point. As the sum of the angles around a point is 360°, the sum of the exterior angles of the above polygons is also 360°.

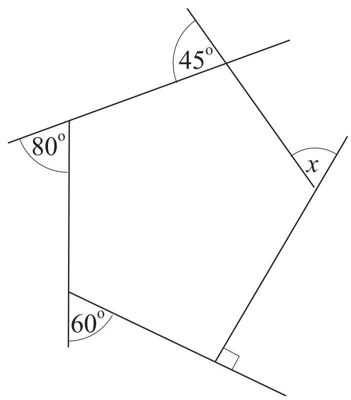
Now let us obtain an expression for the sum of the exterior angles of a polygon which has n sides.
We know that the number of exterior angles and the number of interior angles of a polygon with n sides is n.

At any vertex of a polygon,
the interior angle + the exterior angle $= 180^{\circ}$.
$\therefore$ sum of n interior angles + sum of n exterior angles $= 180^{\circ} \times n$.
But the sum of n interior angles = $(2n - 4)$ right angles = $180°(n - 2)$. Therefore,
$180^{\circ}(n - 2)$ + sum of n exterior angles $= 180^{\circ}n$

$$\begin{aligned}\therefore \text{ sum of } n \text{ exterior angles} &= 180^{\circ}n - 180^{\circ}(n-2)\\ &= 180^{\circ}n - 180^{\circ}n + 360^{\circ}\\ &= 360^{\circ}\end{aligned}$$

Sum of the exterior angles of a polygon = 360°

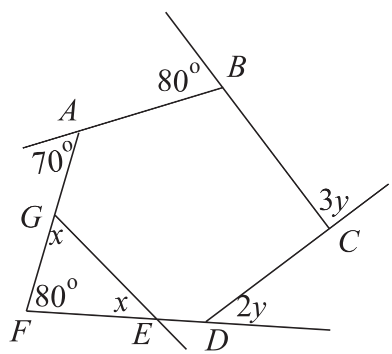
Example 1



Find the magnitude of the exterior angle indicated by x, of the given pentagon.

$$\begin{aligned}\text{Sum of the exterior angles} &= 360^{\circ}\\ \therefore\ x + 45^{\circ} + 80^{\circ} + 60^{\circ} + 90^{\circ} &= 360^{\circ}\\ x + 275^{\circ} &= 360^{\circ}\\ x &= 360^{\circ} - 275^{\circ}\\ x &= \underline{\underline{85^{\circ}}}\end{aligned}$$

Example 2



According to the information marked in the figure,

i. find the value of x

ii. find the value of y.

i. Sum of the interior angles of the triangle EFG $= 180^\circ$

$$\therefore \quad 80^\circ + x + x = 180^\circ$$
$$2x = 180^\circ - 80^\circ = 100^\circ$$
$$x = \frac{100^\circ}{2}$$
$$x = \underline{\underline{50^\circ}}$$

ii. Sum of the exterior angles of the hexagon $ABCDEG$ $= 360^\circ$

$$\therefore \quad 70^\circ + 80^\circ + 3y + 2y + x + x = 360^\circ$$
$$70^\circ + 80^\circ + 5y + 50^\circ + 50^\circ = 360^\circ$$
$$5y = 360^\circ - 250^\circ$$
$$y = \frac{110^\circ}{5}$$
$$y = \underline{\underline{22^\circ}}$$

Example 3

The exterior angles of a quadrilateral are in the ratio 2 : 2 : 3 : 3. Find the magnitude of each exterior angle.

Sum of the exterior angles $= 360^\circ$

Ratio of the 4 angles $= 2 : 2 : 3 : 3$

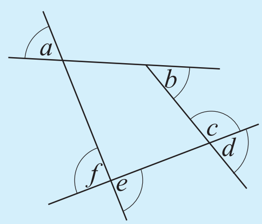
$\therefore$ the smaller angles $= 360^\circ \times \dfrac{2}{10} = 72^\circ$

The larger angles $= 360^\circ \times \dfrac{3}{10} = 108^\circ$

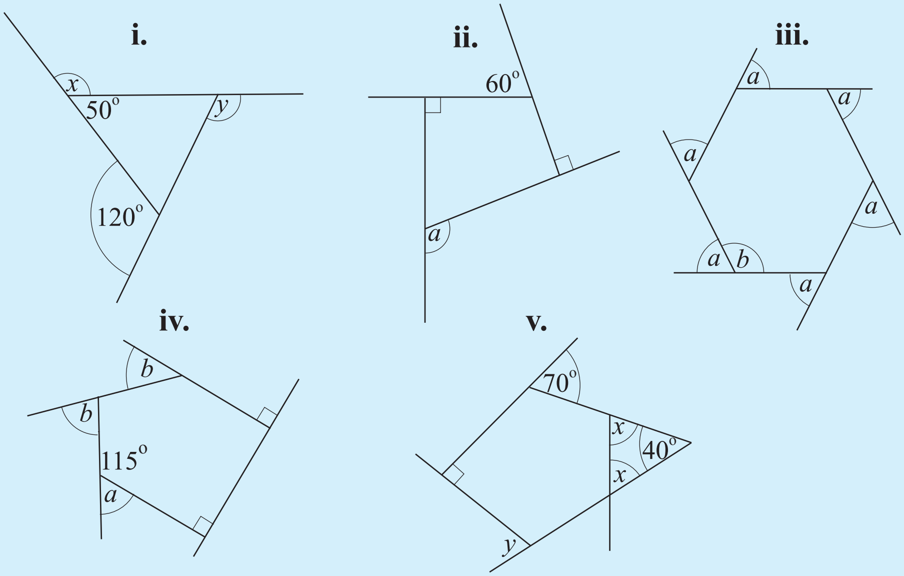
$\therefore$ the exterior angles are $72^\circ, 72^\circ, 108^\circ$ and 108°.

Exercise 25.2

1. From the angles denoted by a, b, c, d, e and f in the figure, select and write the ones which are exterior angles of the quadrilateral.



2. For each of the polygons given below, find the magnitude of the angle/angles denoted by the English letter/letters.



3. The exterior angles of a quadrilateral are x°, $2x^{\circ}$, $3x^{\circ}$ and $4x^{\circ}$.

i. Find the magnitude of each of the exterior angles.

ii. Write the magnitude of each of the interior angles.

4. The exterior angles of a pentagon are in the ratio 1 : 1 : 2 : 3 : 3. Find the magnitude of each of the exterior angles.

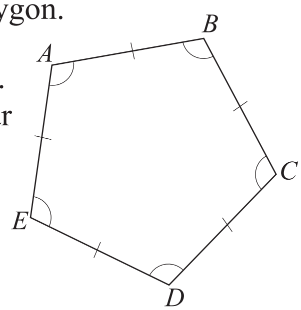
5. The exterior angles of a dodecagon are equal. Find the magnitude of any one of these exterior angles.

6. The magnitude of an exterior angle of a polygon whose exterior angles are equal is 18°. Find the number of sides the polygon has.

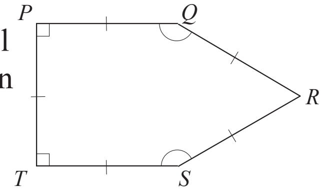
25.3 Regular Polygons

When all the sides of a polygon are equal in length and all the interior angles are equal in magnitude, the polygon is known as a regular polygon.

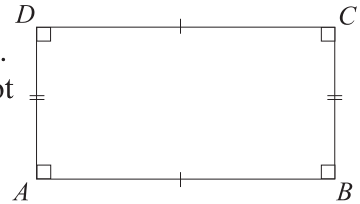
All the sides of the pentagon *ABCDE* in the figure are equal. All the interior angles are also equal. Therefore it is a regular pentagon.



All the sides of the pentagon *PQRST* are equal. However, all the interior angles are not equal. Therefore, the pentagon *PQRST* is not regular.



All the interior angles of the rectangle *ABCD* are equal. However, all the sides are not equal. Therefore, it is not a regular polygon.



Some regular polygons have special names. A regular triangle is called an **equilateral triangle**. A regular quadrilateral is called a **square**.

Example 1

Find the magnitude of an exterior angle of a regular hexagon and thereby find the magnitude of an interior angle.

$$\text{Sum of the six exterior angles} = 360^\circ$$

$$\therefore \text{ the magnitude of an exterior angle} = \frac{360^\circ}{6} = 60^\circ$$

$$\text{exterior angle} + \text{interior angle} = 180^\circ$$

$$\therefore 60^\circ + \text{interior angle} = 180^\circ$$

$$\therefore \text{ interior angle} = 180^\circ - 60^\circ$$

$$= \underline{\underline{120^\circ}}$$

Example 2

The magnitude of an interior angle of a regular polygon is 150°. Find,

i. the magnitude of an exterior angle.
ii. the number of sides.

i. $\text{exterior angle} + \text{interior angle} = 180^\circ$

$$\therefore \text{ exterior angle} + 150^\circ = 180^\circ$$

$$\therefore \text{ exterior angle} = 180^\circ - 150^\circ = \underline{\underline{30^\circ}}$$

ii. $\text{The number of sides} = \dfrac{360^\circ}{30^\circ} = \underline{\underline{12}}$

Exercise 25.3

1. Find the magnitude of an exterior angle of a regular pentagon and thereby find the magnitude of an interior angle.

2. Find the magnitude of an exterior angle of a regular polygon with 15 sides and thereby find the magnitude of an interior angle.

3. **i.** Find the number of sides of a regular polygon with an exterior angle of magnitude 120° and write the special name given to it.

ii. Write with reasons the special name given to the regular polygon which has an exterior angle of magnitude 90°.

iii. Write the name given to the regular polygon which has an exterior angle of magnitude 40°.

4. The magnitude of an interior angle of a regular polygon is four times the magnitude of an exterior angle. Find

i. the magnitude of an exterior angle.

ii. the magnitude of an interior angle.

iii. the number of sides the polygon has.

5. What is the greatest value that an exterior angle of a regular polygon can be? What is the name given to the corresponding regular polygon?

Summary

- Sum of the exterior angles of a polygon = 360°.
- The sum of the interior angles of a polygon of n sides $=$ $(2n - 4)$ right angles.