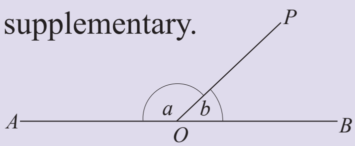
16 Angles of a Triangle

By studying this lesson you will be able to;

- solve simple problems using the theorem "The sum of the interior angles of a triangle is 180^{o}",
- solve simple problems using the theorem "The exterior angle of a triangle is equal to the sum of the interior opposite angles".

Let us recall several results in geometry that you have learnt earlier related to straight lines.

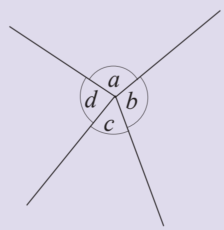
- A pair of adjacent angles on a straight line are supplementary.



AOB is a straight line.

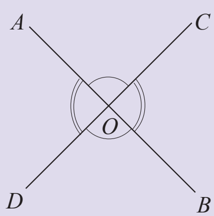
$\therefore\ a + b = 180^{o}$.

- The sum of the angles around a point is 360^{o}.



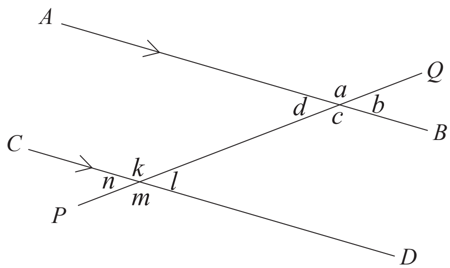
$$a + b + c + d = 360^{o}$$

- The vertically opposite angles formed by the intersection of two straight lines are equal.



AB and CD are straight lines. $A\hat{O}C = B\hat{O}D$ and $A\hat{O}D = C\hat{O}B$.

- Angles related to parallel lines

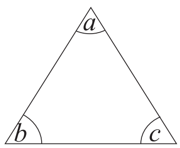


$AB \parallel CD$.

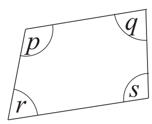
- $c = k$ and $d = l$ (alternate angles)
- $a = k,\ b = l,\ d = n,\ c = m$ (corresponding angles)
- $d + k = 180^{\circ}$ and $c + l = 180^{\circ}$ (allied angles)

In the lesson on triangles and quadrilaterals learnt in grade 8, we identified that;

- the sum of the interior angles of a triangle is 180° and the sum of the interior angles of a quadrilateral is also 360°.

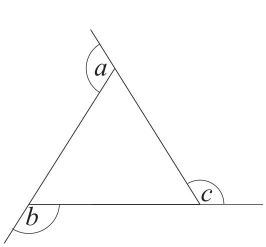


$a + b + c = 180^{\circ}$

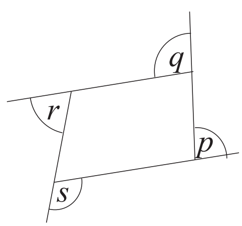


$p + q + r + s = 360^{\circ}$

- the sum of the exterior angles of a triangle is 180° and the sum of the exterior angles of a quadrilateral is also 360°.



$a + b + c = 360^{\circ}$



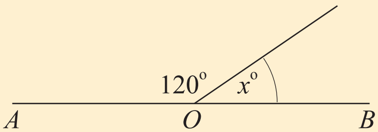
$p + q + r + s = 360^{\circ}$

Do the following review exercise to further establish the above facts.

Review Exercise

a. AOB is a straight line. Find the value of x.

i.



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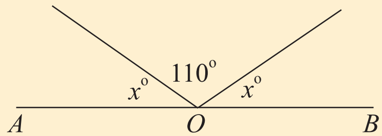
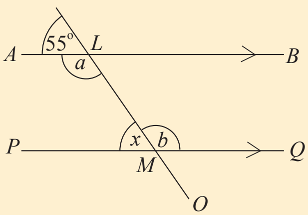
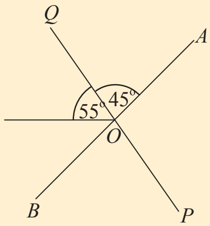


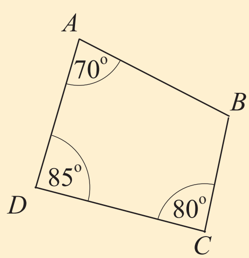
b. Find the magnitude of each of the angles a, b and x, using the information in the figure.



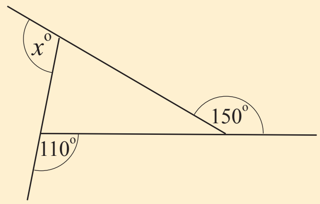
c. AOB and POQ are straight lines. Find the magnitudes of $P\hat{O}B$, $Q\hat{O}B$ and $A\hat{O}P$.



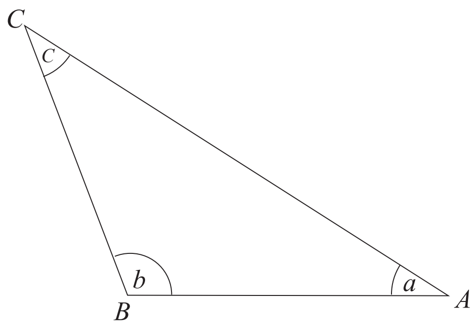
d. Find the magnitude of $A\hat{B}C$ using the information in the figure.



e. Find the value of x, using the information in the figure.



16.1 Interior angles of a triangle



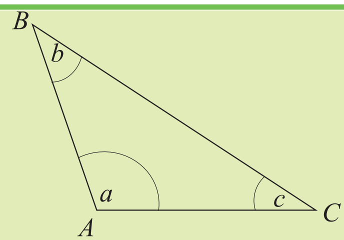
a, b and c are the interior angles of the triangle ABC in the above figure. As discussed earlier, the sum of the interior angles of a triangle is 180°. Hence,

$$A\hat{B}C + B\hat{C}A + C\hat{A}B = 180^{\circ}.$$

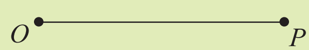
Let us do the following activity to verify the above relationship.

Activity 1

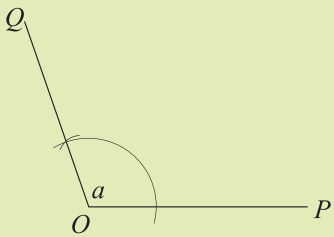
Step 1: Draw a triangle in your exercise book and name it ABC. (The interior angles are given as a, b, c).



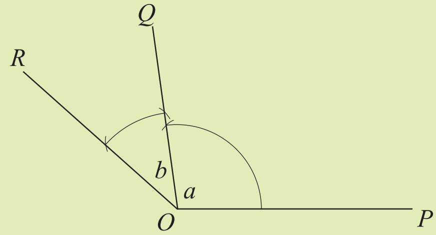
Step 2: Draw a straight line segment in your exercise book and name it OP.



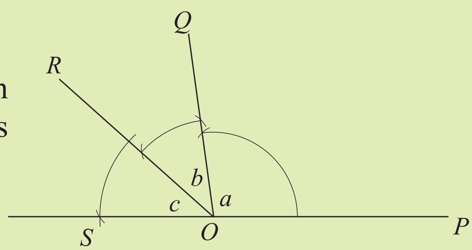
Step 3: Using a straight edge and a pair of compasses, copy the angle $C\hat{A}B$ (a) at O, such that O is the vertex and OP is an arm. (This angle is indicated as $P\hat{O}Q$ in the figure).



Step 4: As done above, copy the angle $A\hat{B}C$ at O, such that O is the vertex and OQ is an arm. (This angle is indicated as $Q\hat{O}R$ in the figure).



Step 5: Copy the angle $A\hat{C}B$ at O, such that O is the vertex and OR is an arm. (This angle is indicated as $R\hat{O}S$ in the figure).



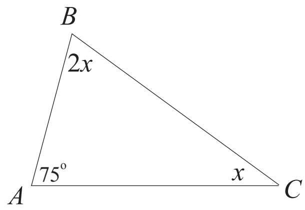
Examine whether *POS* is a straight line by using a straight edge or a protractor, and accordingly write the conclusion we can arrive at.

In the above activity, you must have established that *POS* is a straight line. Since the interior angles of the triangle *ABC* were copied onto the straight line *POS*, and since the sum of the angles on a straight line is 180°, it can be concluded that the sum of the interior angles of the triangle *ABC* is 180°. This can be stated as a theorem as follows.

Theorem: The sum of the three interior angles of a triangle is 180°.

Now let us consider a few examples to see how this theorem can be used to solve problems.

Example 1



Determine the magnitudes of $A\hat{C}B$ and $A\hat{B}C$ of the triangle ABC, using the information given in the figure.

$$
\begin{aligned}
75^{\circ} + 2x + x &= 180^{\circ}\\
3x &= 180^{\circ} - 75^{\circ}\\
3x &= 105^{\circ}\\
x &= \frac{105^{\circ}}{3}\\
&= 35^{\circ}
\end{aligned}
$$

$$\therefore\ A\hat{C}B = x = 35^{\circ}$$

$$A\hat{B}C = 2x = 2 \times 35^{\circ} = 70^{\circ}$$

Example 2

The magnitudes of the interior angles of a triangle are in the ratio 2:3:4. Determine the magnitudes of these angles and giving reasons mention what type of triangle it is.

The ratio of the magnitudes of the angles $= 2:\ 3:\ 4$

$\therefore$ The fractions related to the angles $= \frac{2}{9}, \frac{3}{9}, \frac{4}{9}$

The sum of the three angles $= 180^{\circ}$

$\therefore$ The smallest angle $= \overset{20}{\cancel{180}}{}^{\circ} \times \frac{2}{\underset{1}{\cancel{9}}} = 40^{\circ}$

The medium angle $= 180^{\circ} \times \frac{3}{9} = 60^{\circ}$

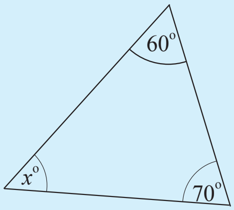
The largest angle $= 180^{\circ} \times \frac{4}{9} = 80^{\circ}$

Hence, the interior angles of the triangle are of magnitudes 40°, 60° and 80°. This is an acute triangle since every angle is less than 90°.

Exercise 16.1

1. Find the magnitude of each angle indicated by a lowercase letter in the following figures, using the information provided in the figures.

i.



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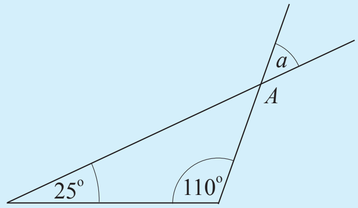

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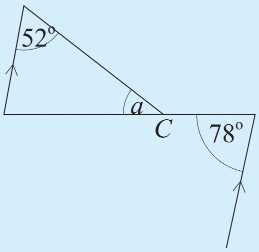

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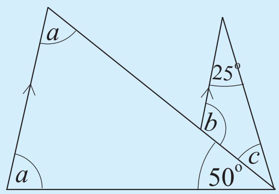
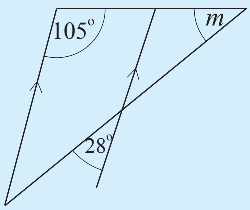
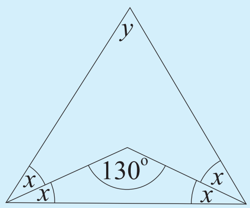


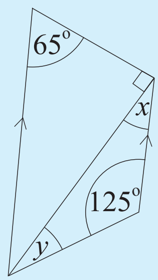
v.



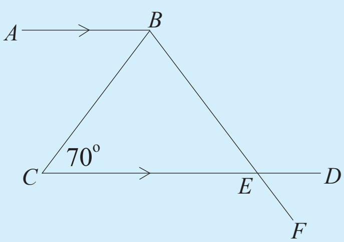
vi.



vii.

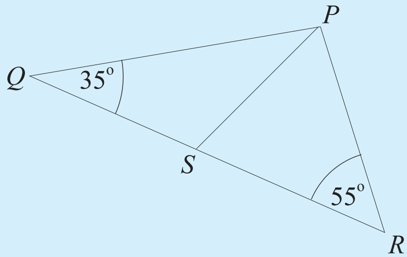


2.



In the given figure, $A\hat{B}C = C\hat{B}E$. $B\hat{C}E = 70^{\circ}$. Find the magnitude of $D\hat{E}F$.

3.



In the triangle PQR, the point S is located on QR such that $Q\hat{P}S = R\hat{P}S$. Moreover, $P\hat{Q}S = 35^{\circ}$ and $P\hat{R}S = 55^{\circ}$.

(i) Find the magnitude of $Q\hat{P}R$.

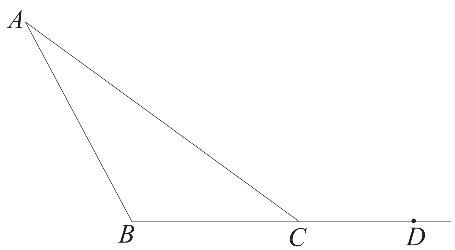
(ii) Find the magnitude of $P\hat{S}R$.

4. In the triangle XYZ, $\hat{X} + \hat{Y} = 115^{\circ}$ and $\hat{Y} + \hat{Z} = 100^{\circ}$. Find the magnitudes of $\hat{X}$, $\hat{Y}$ and $\hat{Z}$.

5. The ratio of the magnitudes of the interior angles of a triangle is 1: 2 : 3. Find the magnitude of each angle separately and with reasons mention what type of a triangle it is.

6. An interior angle of a triangle is 75°. The ratio of the magnitudes of the remaining two angles is 1 : 2. Find the magnitude of each of these angles.

16.2 Exterior angles of a triangle

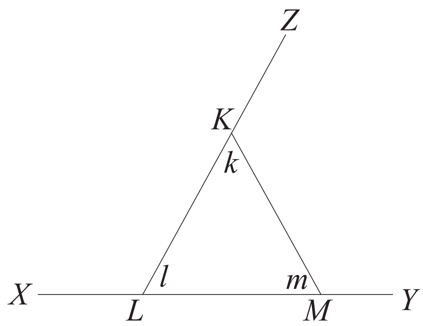


The side BC of the triangle ABC shown in the figure is produced and the point D is marked on BC produced. The angle $A\hat{C}D$ which is formed outside the triangle is called an exterior angle of the triangle.

The interior angle of the triangle, which is adjacent to the exterior angle $A\hat{C}D$ is $A\hat{C}B$. The other two interior angles which are not adjacent to the exterior angle are called the interior opposite angles.

Accordingly, in this figure, the interior opposite angles relevant to the exterior angle $A\hat{C}D$ are $C\hat{A}B$ and $A\hat{B}C$.

Now, let us consider another instance.



In the triangle KLM in the above figure, k, l and m are the interior angles. Three exterior angles have been created by producing the sides of the triangle.

The interior opposite angles relevant to $K\hat{M}Y$ are k and l.

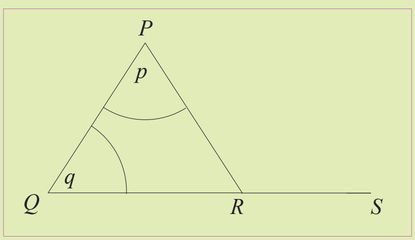
The interior opposite angles relevant to $M\hat{K}Z$ are l and m.

The interior opposite angles relevant to $X\hat{L}K$ are k and m.

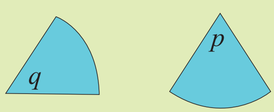
Now let us develop a relationship between an exterior angle and the interior opposite angles of a triangle.

Activity 1

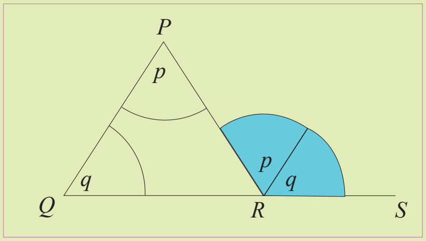
Step 1: Draw a triangle on a piece of Bristol board or on a thick sheet of paper as shown in the figure. Produce a side to create an exterior angle. Mark and shade the interior opposite angles relevant to it (Indicated by p and q in the figure).



Step 2: Using a blade, cut and separate out the interior opposite angles you marked as laminas.



Step 3: Place the two laminas of the interior opposite angles such that they coincide with the exterior angle as shown in the figure, and paste them

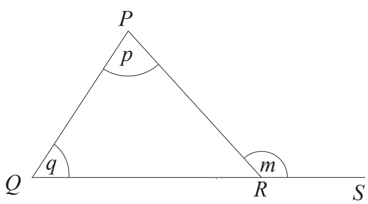


Compare your completed work with those of your friends. Write the conclusion that can be arrived at through this activity.

From the above activity, we can say that the exterior angle of a triangle is equal to the sum of the interior opposite angles.

Draw an acute triangle, a right triangle and an obtuse triangle in your exercise book and in each triangle, mark an exterior angle and the relevant interior opposite angles. Measure them using a protractor and verify the above relationship for the three triangles by obtaining the sum of the interior opposite angles.

The above result can be expressed as follows.



$m = p + q.$

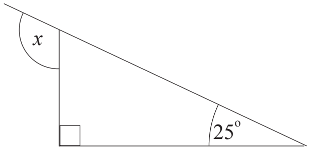
That is, $P\hat{R}S = R\hat{P}Q + P\hat{Q}R$.

This can be expressed as a theorem.

Theorem: **If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.**

Now, let us consider a few examples to see how this result can be used to solve problems.

Example 1



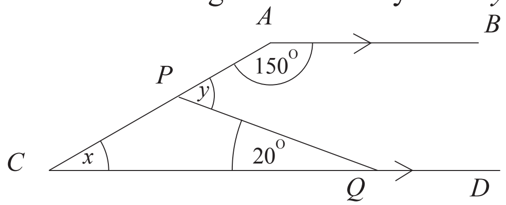
Find the magnitude of the angle indicated by x in the figure.

$x = 90^\circ + 25^\circ$

$\quad = \underline{\underline{115^\circ}}$

Example 2

Find the magnitudes of the angles denoted by x and y in the figure.



$x + 150^\circ = 180^\circ$ (since $AB \parallel CD$ and allied angles are supplementary)

$x = 180^\circ - 150^\circ = 30^\circ$

$y = x + 20^\circ$ (the exterior angle of triangle PCQ = the sum of the interior opposite angles)

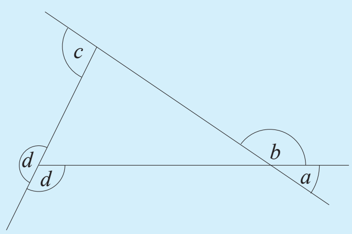
$y = 30^\circ + 20^\circ$

$\quad = \underline{\underline{50^\circ}}$

Exercise 16.2

1. Select and write the letters corresponding to the angles which are exterior angles of the given triangles.

i.



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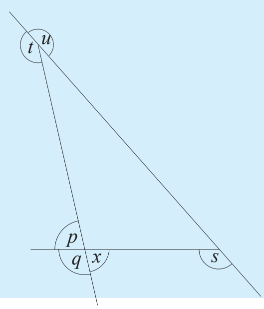
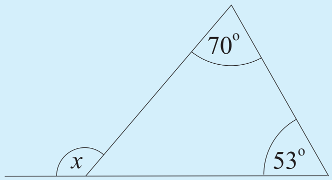


2. Find the magnitude of each angle denoted by a lowercase letter in the following figures.

i.



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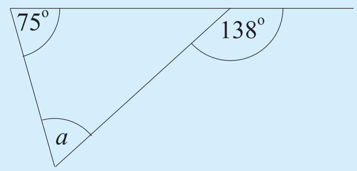

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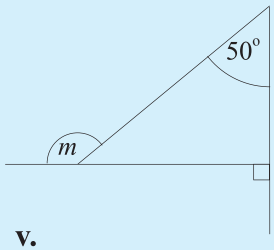

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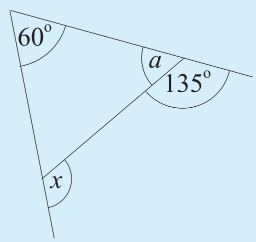

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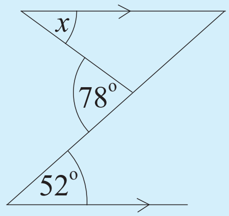

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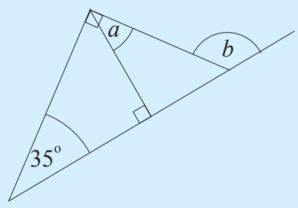

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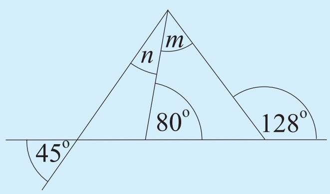
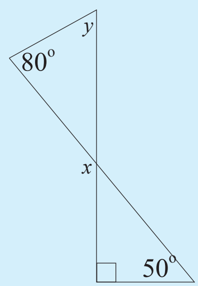
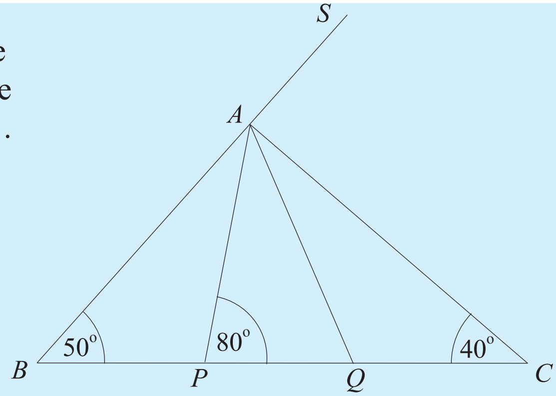


viii.

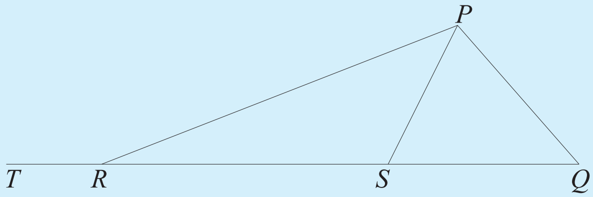


3. In the triangle ABC in the figure, the points P and Q are located on the side BC such that $B\hat{A}P = C\hat{A}Q$. The side BA is produced to S.

i. Find the magnitude of $B\hat{A}P$.
ii. Find the magnitude of $A\hat{Q}P$.
iii. Find the magnitude of $S\hat{A}Q$.

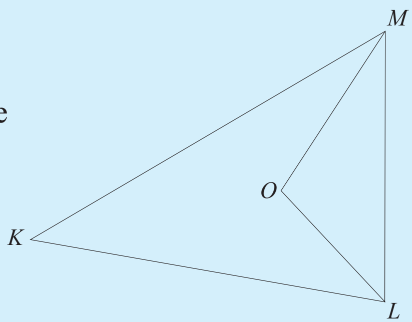


4. In the triangle PQR shown in the figure, the bisector of $\hat{P}$ meets QR at S. Moreover, $S\hat{P}Q = S\hat{Q}P$. If $S\hat{Q}P = a^{\circ}$, then find $P\hat{R}T$ in terms of a.

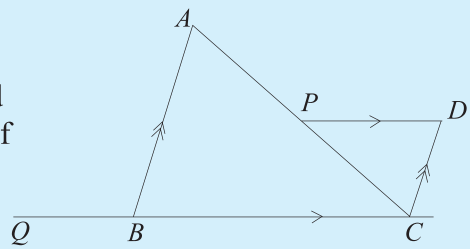


Miscellaneous Exercise

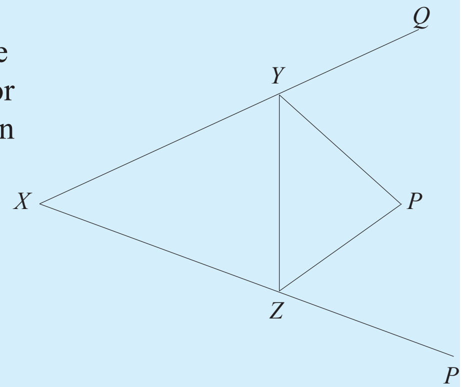
1. In the triangle KLM, the angle bisectors of $\hat{M}$ and $\hat{L}$ meet at O. Moreover, $\hat{K} = 70^{\circ}$. Find the magnitude of $L\hat{O}M$.



2. In the given figure, $A\hat{P}D = 140^{\circ}$ and $P\hat{D}C = 85^{\circ}$. Find the magnitude of $A\hat{B}Q$.



3. The sides XY and XZ of the triangle XYZ have been produced. The bisectors of the exterior angles at Y and Z intersect at P. Find $Y\hat{P}Z$ in terms of $\hat{X}$.



Summary

- **The sum of the three interior angles of a triangle is 180°.**
- **If a side of a triangle is produced the exterior angle so formed is equal to the sum of the two interior opposite angles.**