14 Loci and Constructions

By studying this lesson you will be able to,

- identify four basic loci,
- construct a line perpendicular to a given line,
- construct the perpendicular bisector of a straight line segment,
- construct and copy angles,
- solve problems related to loci and constructions.

Loci

A few motions that you can observe in the environment are given below. Let us consider the path of each motion.

1. Cotton floating in the air
2. A bird flying
3. A ball hit by a bat
4. A fruit falling from a tree
5. The tip of a hand of a working watch
6. A child riding a see-saw

You may observe that even though the motions of 1 and 2 are complex and unpredictable, the motions of 3 to 6 have a definite path. It is important to learn about loci in geometry to develop a proper understanding of the paths of the objects undergoing these motions.

A set of points satisfying one or more conditions is known as a locus.

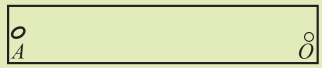
14.1 Basic Loci

Now let us consider four basic loci.

1. The locus of points which are at a constant distance from a fixed point.

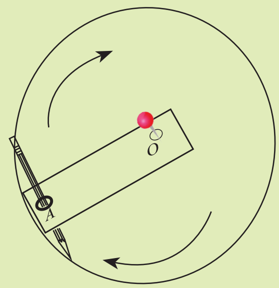
Activity 1

Step 1: Take a strip of cardboard of length 5 cm, make two small holes near the two ends and name them O and A.



Step 2: Keep the above strip of cardboard on a piece of paper, place a pin through the hole O and keep it firm on the piece of paper.

Step 3: Place a pencil point through the hole A and while holding the pin tightly so that it doesn't move, move the pencil and mark the path it takes.



Step 4: At the end of the activity, identify the locus that is obtained.

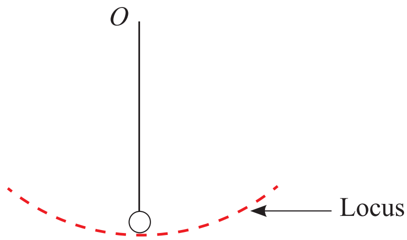
In the above activity, you would have obtained a circular path. Accordingly,

The locus of points on a plane which are at a constant distance from a fixed point is a circle.

Example 1

Draw a sketch of the locus of the bottommost point of the bob of a pendulum in a working pendulum clock.

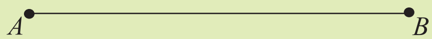
The locus relevant to this motion is a part of a circle with centre the fixed point of the rod/string to which the bob is connected and radius the distance from this fixed point to the bottommost point of the bob.



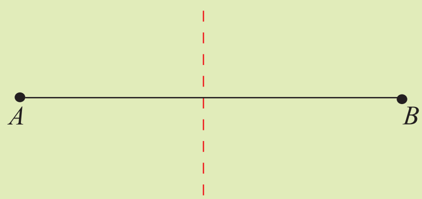
2. The locus of points which are equidistant from two fixed points.

Activity 2

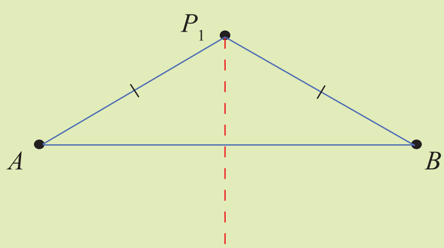
Step 1: Draw a straight line segment of length 10 cm on an oil paper/tissue paper and name it AB.



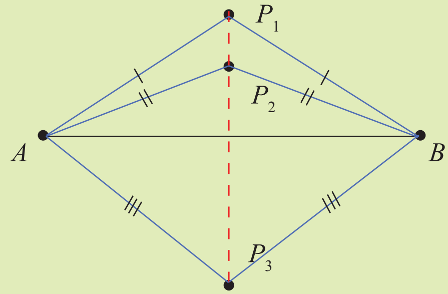
Step 2: Identify the axis of symmetry of the straight line AB by folding the tissue paper such that the two points A and B coincide and mark it with a dashed line.



Step 3: Mark a point P_1 on the dashed line, draw the straight lines P_1A and P_1B, and measure and write their lengths.



Step 4: Mark several more points on the dashed line and measure and write the distances from the points A and B to each of these points.



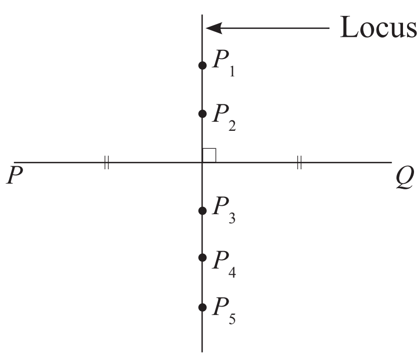
Step 5: Check whether the distances from the points A and B to each of these points on the dashed line are equal and write your conclusion.

When folding the paper as above, such that A and B coincide, observe that the fold line obtained is perpendicular to AB and that it passes through the midpoint of AB. This line is called the perpendicular bisector of the line segment AB. Observe further that the distances from the points A and B to any point you chose on the perpendicular bisector are equal.

The locus of points which are equidistant from two given points is the perpendicular bisector of the line joining the two points.

Example 2

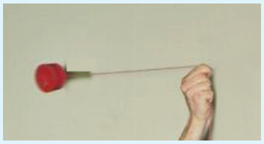
Draw a rough sketch of the locus of points which are equidistant from the two given points P and Q. Name five points on the locus as P_1, P_2, P_3, P_4 and P_5.



Exercise 14.1

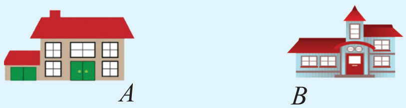
1. By a rough sketch, show the locus relevant to each motion given below.

a. The path of a rubber bushing which is tied to one end of a rope of length 50 cm and rotated, by holding the stretched rope at the other end as shown in the figure



b. The path of the tip of a hand of a working clock

c. The figure shows two houses located 50 m from each other on horizontal ground. It is required to build a wall exactly halfway between the two houses (Points A and B). Indicate by a rough sketch where the wall should be built.



c. The path of the fire of a torch held by a fire torch rotating dancer in a perahera (while the fire torch rotator is stationery)

d. The path of a person riding a Ferris wheel

e. The paths of the two children riding a see-saw while sitting at the two ends of the see-saw

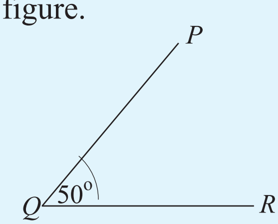
2. In the given figure, P and Q are two trees planted on flat ground, at a horizontal distance of 25 m from each other.

   i. It is required to fix a tap at a distance of 15 m from each tree. With the knowledge on loci, draw a rough sketch to indicate how the points where the tap can be fixed are found.

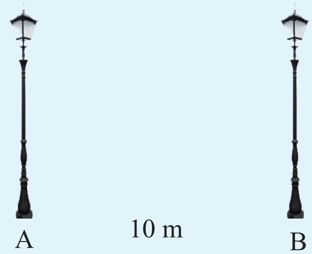
   ii. It is required to cut a drain equidistant from the trees. Draw a rough sketch to indicate the location of the drain.

3. Draw an angle of 50° and name it $P\hat{Q}R$ as shown in the figure.

   With the knowledge on loci, draw a rough sketch to indicate how the point which is equidistant from the points Q and R and lying on the arm PQ is found, and name it S.



4. A and B are two lamp posts located at a distance of 10 m from each other.



   i. It is required to fix another lamp post C at a distance of 6 m from A and 8 m from B. Mark the location of the lamp post C on a suitable rough sketch.

   ii. It is required to fix a lamp post D equidistant form the posts A and B. Mark the location of the lamp post D on a suitable rough sketch.

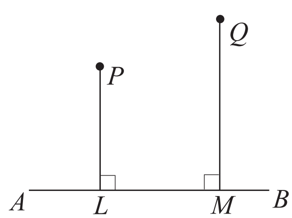
14.2 More on basic loci

3. The locus of points which are at a constant distance from a fixed line

The distance from a point to a line is the length of the perpendicular line drawn from the point to the line.

Accordingly,

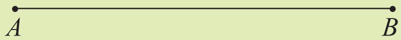
the distance from P to the line AB is the length of PL and the distance from Q to the line AB is the length of QM.



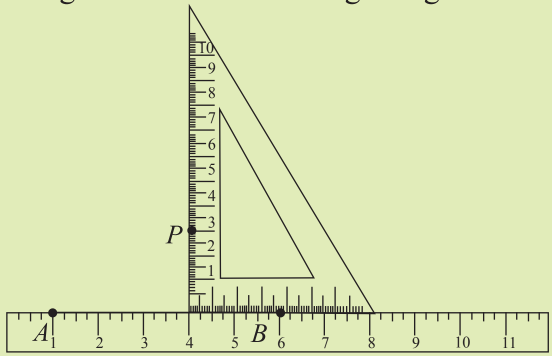
Now let us do the following activity to determine the locus of points which are at a constant distance from a fixed line.

Activity 1

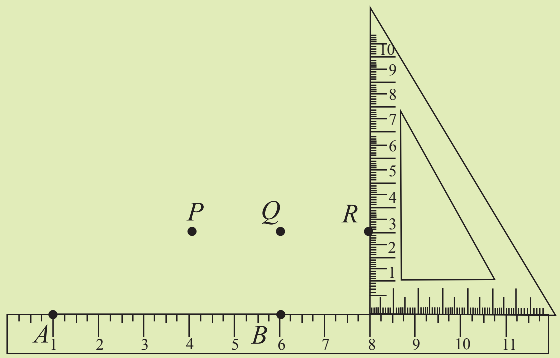
Step 1: Draw a straight line segment in your exercise book and name it *AB*.



Step 2: Place a straight edge on the line *AB* and a set square touching the straight edge as shown in the figure. Mark a point 3 cm from *AB* using the scale on the straight edge and name it *P*.



Step 3: By changing the position of the set square, mark a couple more points at a distance of 3 cm from *AB* and name them *Q* and *R*.



Step 4: Using a straight edge, join the above marked points *P*, *Q* and *R*.

Step 5: Describe the locus of points which are at a distance of 3 cm from the line *AB*. Observe that a similar locus can be drawn on the other side of *AB* too.

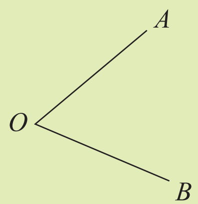
It is clear from the above activity that, the locus of a point at a constant distance of 3 cm from the line AB is a straight line parallel to the line AB and at a distance of 3 cm from it. Moreover, two loci can be drawn on either side of AB.

The locus of points which are at a constant distance from a straight line are the two straight lines parallel to it and at the given constant distance from it, on either side of it.

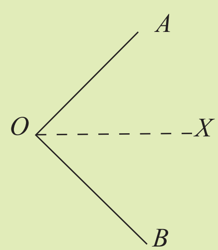
4. The locus of points equidistant from two intersecting straight lines.

Activity 2

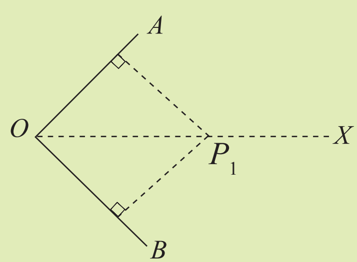
Step 1: On a transparent paper (like oil paper) draw a pair of straight lines as shown in the figure and name them OA and OB.



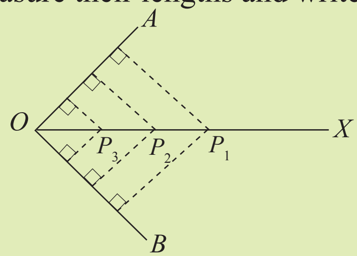
Step 2: Fold the transparent paper such that OA and OB coincide, and mark the fold line with a dotted line. Name it OX.



Step 3: Mark a point on the dotted line and name it P_1. Using a set square, draw two lines from P_1 perpendicular to OA and OB respectively, measure their lengths and write them down.



Step 4: Mark more points on line OX as shown in the figure and name them P_2, P_3, etc. From each of these points, draw perpendicular lines to OA and OB, measure their lengths and write them down.



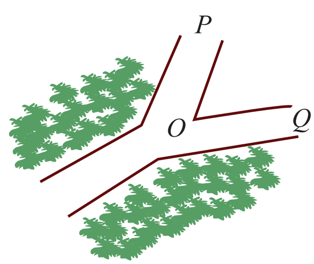
Step 5: Measure $A\hat{O}X$ and $B\hat{O}X$ and write what can be concluded about the line OX.

From the above activity it is clear that OX is the line that divides the angle $A\hat{O}B$ into two equal angles and that the distances from any point on the line OX to the lines OA and OB are equal.
Furthermore, since the paper was folded such that OA and OB coincide, the angles $A\hat{O}X$ and $B\hat{O}X$ are equal to each other.

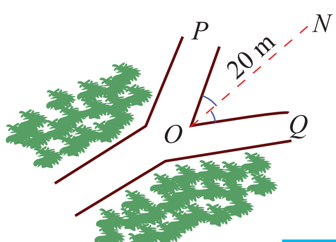
OX is known as the angle bisector of $A\hat{O}B$.

The locus of points equidistant from two intersecting straight lines is the angle bisector of the angles formed by the intersection of the two lines.

Example 1



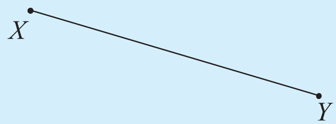
OP and OQ are two roads which diverge at the junction O. It is required to fix a notice board at a point which is 20 m from the junction O and at an equal distance from both these roads. Using the knowledge on loci, indicate by a rough sketch how you would find the place where the notice board should be fixed.



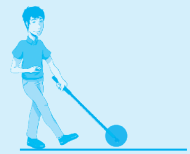
The required position (say N) should be a point on the angle bisector of $Q\hat{O}P$. Since $ON = 20$ m, N should be the point on the angle bisector a distance of 20 m from O.

Exercise 14.2

1. Draw a straight line segment and name it XY. Illustrate by a rough sketch, the locus of points which are 4 cm away from it.



2. A student walks on a straight road rotating a wheel of diameter 20 cm which is fixed to a handle. Illustrate by a rough sketch the locus of the centre of the wheel.

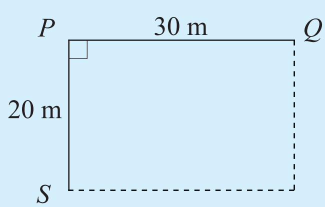


3. The figure shows the positions of the hour hand and the minute hand of a clock at a certain instant. At this moment, the second hand is located at an equal distance from these two hands. Indicate the position of the second hand by a separate rough sketchs.

4. *A* drain PQ of length 50 m which is located in a certain plot of land is shown in the figure. A tap needs to be fixed at a distance of 10 m from PQ and at an equal distance from both the ends P and Q. Illustrate by a rough sketch the position/ positions where the water tap can be fixed.

5. A piece of cake cut from a round (circular) cake is shown in the figure. It is required to divide this piece of cake into two equal pieces. Using the knowledge on loci, indicate by a sketch how this piece should be cut.

6. PQ and PS are two boundaries of a rectangular plot of land. A tree needs to be planted in this plot of land such that it is 8 m from the boundary PQ and 5 m from the boundary PS. Illustrate by a rough sketch where the tree should be planted and name it T.



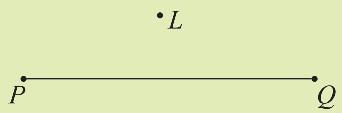
14.3 Constructing lines perpendicular to a given straight line

Let us explain two phrases that are commonly used in constructions. When drawing a circle using a pair of compasses, phrases such as "taking a certain point as the centre" and "taking a certain length as the radius" are often used. For example, "Taking point A as the centre" means the circle or arc should be drawn with the point of the pair of compasses kept at the point A; and "Taking AB as the radius" means that the distance between the point of the pair of compasses and the pencil point should be equal to the length of AB.

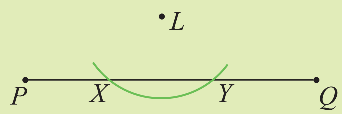
1. Constructing a line perpendicular to a given line from an external point

Activity 1

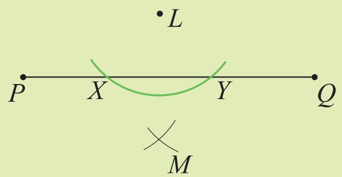
Step 1: Draw a straight line segment in your exercise book and name it PQ. Mark a point external to PQ and name it L.



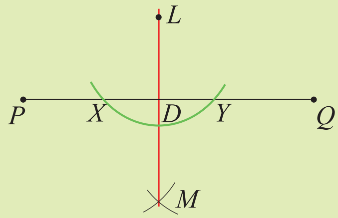
Step 2: Taking a length which is more than the distance from L to PQ as the radius and L as the centre, drawn arc such that it intersects the line PQ. Name the points of intersection X and Y.



Step 3: Taking each of the points X and Y as the centre and using the same radius, draw two arcs such that they intersect each other as shown in the figure. Name the point of intersection M.



Step 4: Join the points L and M and name the point at which LM intersects PQ as D. Measure and write the magnitude of $L\hat{D}P$.



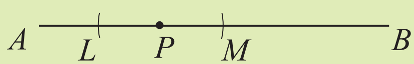
At the end of the above construction, you would have obtained that $L\hat{D}P = 90^{o}$. That is, LD is the perpendicular line drawn from the point L to the line PQ.

2. Constructing a line perpendicular to a given line through a point on the line

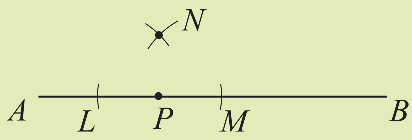
Activity 2

Step 1: Draw a straight line and name it AB. Mark a point on it and name it P.

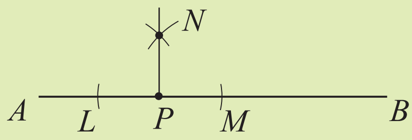
Step 2: Taking a length less than the length of PA as the radius, and taking P as the centre, draw two arcs using the pair of compasses such that they intersect the line segments PA and PB. Name the two points of intersection L and M.



Step 3: Taking a length greater than the one taken in step 2 as the radius, and taking L and M as the centres, draw two arcs such that they intersect each other as shown in the figure. Name the point of intersection N.



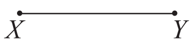
Step 4: Join *NP*, measure the magnitude of the angle $N\hat{P}A$ and write its value.



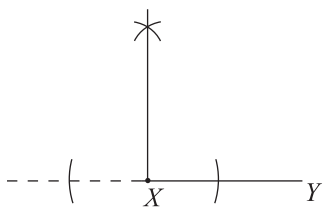
At the end of the above construction you would have obtained that $N\hat{P}A = 90^{\circ}$. That is, the line drawn perpendicular to *AB* through the point *P* is *PN*.

3. Constructing a line perpendicular to a given straight line segment through an end point

Let us assume that we need to draw a line perpendicular to the line segment *XY* through the point *X*.



Produce the line *YX* and do this construction using the method identified above.

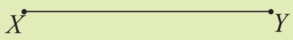


4. Constructing the perpendicular bisector of a straight line segment

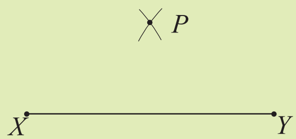
The straight line which is perpendicular to a given line segment and which passes through the midpoint of that line segment, was identified earlier as the perpendicular bisector of that line segment.

Draw a straight line segment and name it *XY*. Let us do the activity given below to construct the perpendicular bisector of this line segment

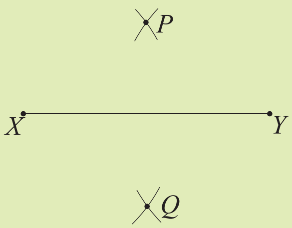
Activity 3



Step 1: Taking a length greater than half the length of XY as the radius, and without changing it, draw two arcs with X and Y as the centres, such that they intersect each other. Name the point of intersection P.

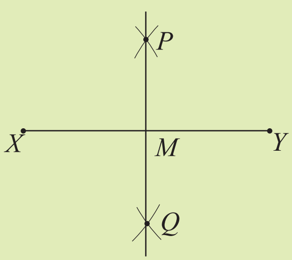


Step 2: As done above, taking X and Y as the centres, draw two other arcs such that they intersect each other on the side of XY opposite to the side on which P is located. Name the point of intersection Q.



Note: It is not necessary to use the same radius in the above two steps.

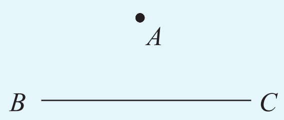
Step 3: Join PQ and name the point at which PQ intersects XY as M. Measure XM and MY and the magnitude of $X\hat{M}P$. What can be concluded regarding the line PQ?



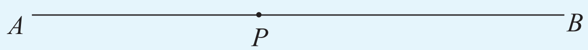
You would have identified in the above activity that $XM = MY$ and $X\hat{M}P = 90^{\circ}$. Accordingly, PQ bisects the line segment XY perpendicularly. Therefore, PQ is the perpendicular bisector of XY.

Exercise 14.3

1. Draw a straight line as shown in the figure and name it BC. Construct a perpendicular line from the point A to the line BC.



2. Draw the line AB such that $AB = 7$ cm. Mark the point P on AB such that $AP = 3$ cm and construct a line perpendicular to AB through P.



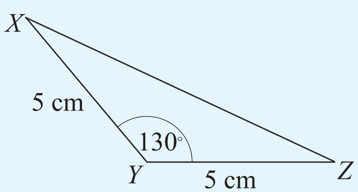
3. Draw any acute angled triangle and name it PQR.

i. Construct a line perpendicular to QR from P.

ii. Construct a line perpendicular to PR from Q.

iii. Construct a line perpendicular to PQ from R.

4. i. Using a protractor, draw an angle of 130° and as shown in the figure mark 5 cm on each arm and complete the triangle XYZ.



ii. Construct a perpendicular line from Y to the line XZ and name the point at which it meets XZ as D.

iii. Measure and write the lengths of XD and ZD.

5. Construct a rectangle of length 6 cm and breadth 4 cm.

6. a. Draw a straight line segment PQ such that $PQ = 10$ cm.

b. Mark the point B on the line PQ such that $PB = 2$ cm.

c. Construct a line perpendicular to PQ through B.

d. Mark a point A on the perpendicular line such that $BA = 6$ cm and complete the triangle ABQ.

e. Construct the perpendicular bisector of the line segment *BQ* and name the point it intersects *AQ* as *O*.
f. Construct a circle with *O* as the centre and *OA* as the radius.

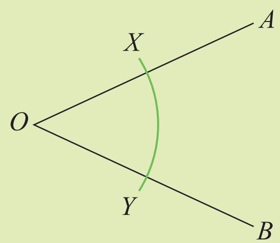
14.4 Constructions related to angles

Constructing the angle bisector

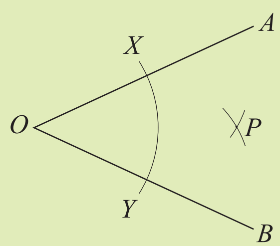
The line drawn through a given angle such that it divides the angle into two equal angles, is known as the angle bisector of the given angle.

Activity 1

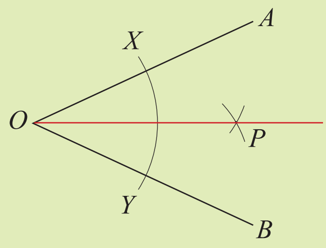
Step 1: Draw an arc with *O* as the centre such that it intersects the arms *OA* and *OB*. Name the points of intersection *X* and *Y*.



Step 2 : Using a pair of compasses and taking a suitable radius, construct two arcs with *X* and *Y* as the centres such that they intersect each other as shown in the figure. Name the point of intersection *P*.



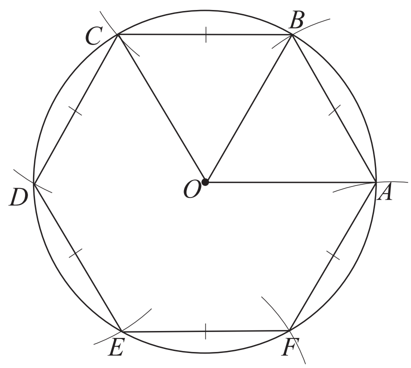
Step 3 : Join OP. Measure $A\hat{O}P$ and $B\hat{O}P$ and check whether they are equal.



It must have been clear to you at the end of the above activity that $A\hat{O}P = B\hat{O}P$. That is, OP is the angle bisector of $A\hat{O}B$.

14.5 Construction of angles

By now we have learnt to draw angles using the protractor. However we can construct a few special angles by using a straight edge and a pair of compasses only. Let us recall how we constructed a regular hexagon in grade 8 by using a pair of compasses.



Here, taking the length of a side of the regular hexagon which needs to be drawn as the radius, a circle is drawn, and with the same radius, arcs are marked on it.The points at which the arcs intersect the circle are joined to each other and to the centre as shown in the figure.

Then every angle of each equilateral triangle that is formed is 60°.

Therefore, $A\hat{O}B = 60^{\circ}$ and $A\hat{O}C = 120^{\circ}$.

Let us use the principles that were used in this construction to construct certain special angles.

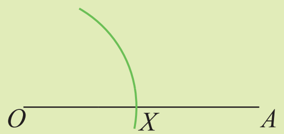
1. Constructing an angle of 60°

Activity 1

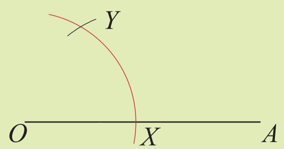
Suppose we need to construct an angle of 60°at O with OA as an arm.

Step 1: Draw a straight line segment in your exercise book and name it OA.

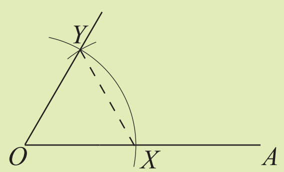
Step 2: Taking O as the centre, construct an arc such that it intersects OA as shown in the figure. Name the point of intersection X.



Step 3: Without changing the length of the radius, and taking X as the centre, draw another arc using the pair of compasses, such that it intersects the first arc. Name this point of intersection Y.



Step 4: Join the points O and Y and produce it as required. Measure $A\hat{O}Y$ and check whether it is 60°.



The triangle OXY in the above figure is an equilateral triangle. The reason for this can be explained as follows.

Since OX and OY are radii of the circle with centre O, $OX = OY$.
Similarly, since XO and XY are radii of the circle with centre X, $XO = XY$.
Accordingly, $OX = XY = OY$.
Therefore, OXY is an equilateral triangle.
Therefore, every angle of it is 60°.

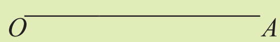
Therefore $X\hat{O}Y = 60^{o}$.

Suppose we need to construct an angle of 120° at O with OA as an arm.

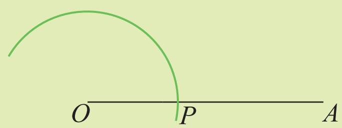
2. Constructing an angle of 120°

Activity 2

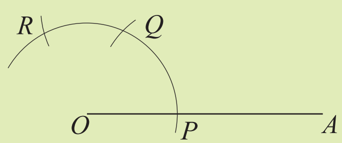
Step 1: Construct a straight line segment and name it OA.



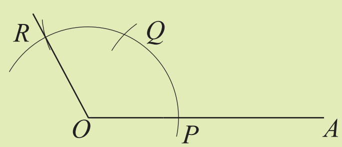
Step 2: Taking O as the centre, construct an arc such that it intersects OA as shown in the figure. Name the point of intersection P.



Step 3: Without changing the length of the radius, and taking P as the centre, draw a small arc using the pair of compasses, such tha t it intersects the first arc as shown in the figure, and name that point of intersection Q. Now, without changing the radius, take Q as the centre and draw another small arc such that it too intersects the first arc and name that point of intersection R.



Step 4: Join OR and produce it as required. Measure and check the magnitude of $A\hat{O}R$.



The reason why $A\hat{O}R = 120^{\circ}$ is the following. As discussed above, $P\hat{O}Q = 60^{\circ}$. Furthermore, QOR is also an equilateral triangle. Therefore, $Q\hat{O}R = 60^{\circ}$. Accordingly,

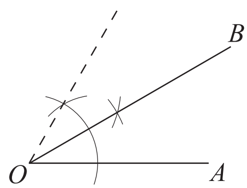
$$\begin{aligned} P\hat{O}R &= P\hat{O}Q + Q\hat{O}R \\ &= 60^{\circ} + 60^{\circ} \\ &= 120^{\circ} \end{aligned}$$

3. Constructing angles of 30°, 90° and 45°

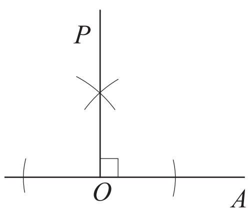
By constructing suitable angle bisectors we can construct the angles 30°, 90° and 45°. By considering the information and figures given below construct the given angles.

Angle of 30°

Construct an angle of 60° and construct its angle bisector. Then $A\hat{O}B = 30^{\circ}$.

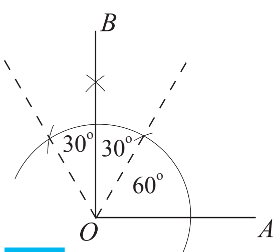


Angle of 90°



Method I

At O, construct a line perpendicular to the line segment AO. Then $A\hat{O}P = 90^{\circ}$.



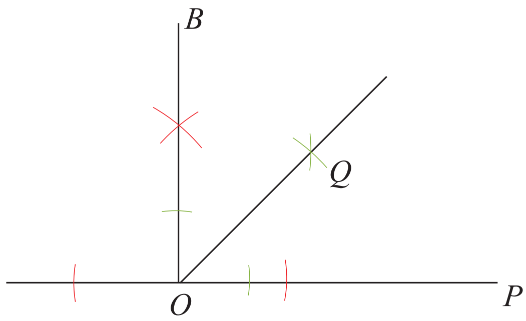
Method II

Construct an angle of 120°and bisect one 60° angle. Then $A\hat{O}B = 90^{\circ}$.

Angle of 45°

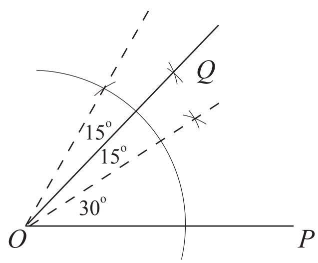
Method I

Construct an angle of 90° and bisect it. Then $P\hat{O}Q = 45^{\circ}$.



Method II

Construct an angle of 60° and bisect it. Again bisect one of the resulting 30° angles. Then, $P\hat{O}Q = 30^{\circ} + 15^{\circ} = 45^{\circ}$.



Copying a given angle

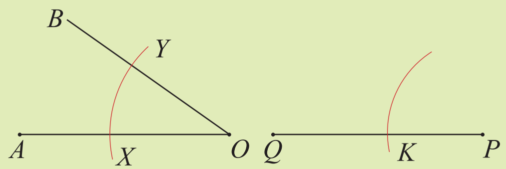
Let us suppose that we need to construct an angle equal to a given angle $A\hat{O}B$ at a point *P*, with *PQ* as an arm. For this, let us do the following activity.

Activity 3

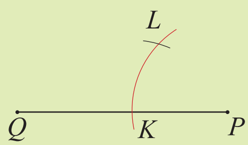
Step 1: Draw any angle and name it $A\hat{O}B$. Draw the arm PQ on which $A\hat{O}B$ needs to be copied.

Step 2: Taking O as the centre, draw an arc as shown in the figure such that it intersects the arms OA and OB, and name the points of intersection X and Y. Using the same radius and taking P as the centre, draw an arc longer than the previous arc such that it intersects PQ.

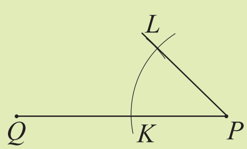
Name the point at which the arc intersects PQ as K.



Step 3: Taking XY as the length of the radius and K as the centre, using the pair of compasses, construct a small arc such that it intersects the initial arc and name the point of intersection L.



Step 4: Join PL and produce it as required. Using a protractor (or any other method), check whether $A\hat{O}B$ and $Q\hat{P}L$ are equal.



Exercise 14.4

1. **i.** Draw a straight line segment of length 8 cm and name it PQ.
   ii. Construct an angle of 60°at P such that PQ is an arm.
   iii. Construct an angle of 60°at Q such that QP is an arm.

2. **i.** Draw a straight line segment of length 6.5 cm and name it AB.
   ii. Construct an angle of 90° at A such that AB is an arm.

iii. Construct an angle of 30°at B such that BA is an arm.

iv. Produce the constructed lines so that they intersect. Name their point of intersection as C and form the triangle ABC.

3. Construct angles of magnitude 15° and 75°.

4. To construct the triangle shown in the figure below, do the following constructions.

i. Draw a straight line segment of length 7 cm and name it PQ.

ii. Construct an angle of 30° at P such that PQ is an arm.

iii. Construct an angle of 45° at Q such that QP is an arm.

iv. Complete the triangle PQR and measure and write the magnitude of $P\hat{R}Q$.

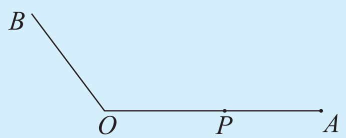
5.

i. Draw a straight line segment OA of length 10 cm.

ii. Draw an arm BO such that $A\hat{O}B$ is an obtuse angle.

iii. Mark the point P on OA such that $OP = 7$ cm.

iv. Construct a line segment PC such that C is on the same side of OA as B and such that $A\hat{P}C = A\hat{O}B$.



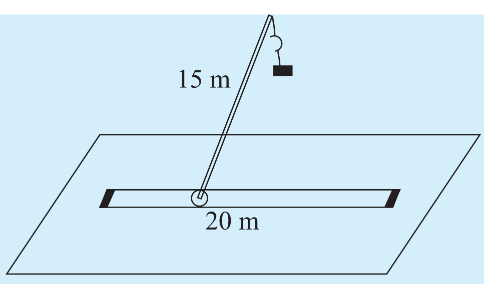
6. **i.** Draw any acute angle and name it $K\hat{L}M$.

ii. Copy the angle $\hat{L}$ at M such that $K\hat{L}M = L\hat{M}N$, where N is on the same side of LM as K.

iii. Name the point of intersection of the lines LK and MN as P (produce the lines if necessary) and measure and write the lengths of PL and PM.

Miscellaneous Exercise

1. In a factory, a 15 m long arm of a crane is fixed to a groove of length 20 m. It can be moved along the groove and also rotated in a horizontal plane about the end points of the groove. Draw a rough sketch, and indicate with measurements the path on the horizontal plane where the crane can exchange goods.



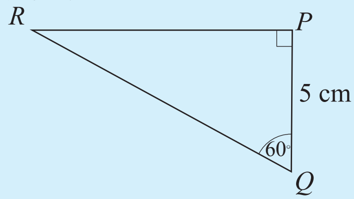
2. To construct the triangle shown in the figure, carry out the steps given below.

i. Draw a straight line segment PQ where $PQ = 5$ cm.

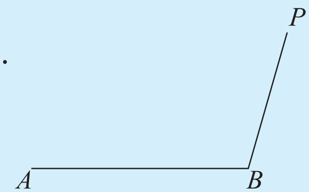
ii. Construct an angle of $90°$ at P.

iii. Construct an angle of $60°$ at Q.

iv. Complete the triangle PQR and measure and write the magnitude of $\hat{R}$.



3. i. As shown in the figure, draw an obtuse angle $A\hat{B}P$.



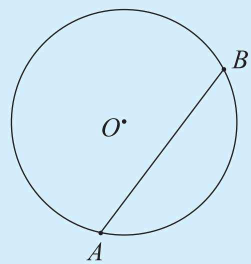
ii. Locate a point K such that $A\hat{B}P = B\hat{P}K$ and such that the two angles form a pair of alternate angles. Join PK.

4. **i.** Draw a circle of radius 4 cm and name its centre O.

ii. Mark two points A and B on the circle 6 cm apart from each other, and draw the line AB.

iii. Construct a perpendicular line from O to AB and name the point at which it meets AB as N.

iv. Measure and write the lengths of AN and BN.



Summary

A set of points satisfying one or more conditions is known as a locus.

Basic Loci

- **The locus of points on a plane which are at a constant distance from a fixed point is a circle.**
- **The locus of points which are equidistant from two given points is the perpendicular bisector of the line joining the two points.**
- **The locus of points which are at a constant distance from a straight line are the two straight lines parallel to it and at the given constant distance from it, on either side of it.**
- **The locus of points equidistant from two intersecting straight lines is the angle bisector of the angles formed by the intersection of the two lines.**