12 Indices

By studying this lesson you will be able to;

- identify the laws of indices on the product of powers, the quotient of powers and the power of a power,
- simplify algebraic expressions using the above mentioned laws of indices,
- identify the zero index and negative indices and simplify algebraic expressions containing these.

Indices

You have learnt about powers of numbers such as 2^1, 2^2 and 2^3 in previous grades. The values of these powers can be obtained as follows.

$$2^1 = 2$$
$$2^2 = 2 \times 2 = 4$$
$$2^3 = 2 \times 2 \times 2 = 8$$
$$\vdots \qquad \vdots$$

You have also learnt about powers of algebraic symbols such as x^1, x^2 and x^3 .These can be expanded and written as follows.

$$x^1 = x$$
$$x^2 = x \times x$$
$$x^3 = x \times x \times x$$
$$\vdots \qquad \vdots$$

Moreover, you have learnt how to write the expanded form of a product of powers of algebraic symbols and numerical values. For example, $5^2a^3b^2$ is written in expanded form as follows.

$$5^2a^3b^2 = 5 \times 5 \times a \times a \times a \times b \times b.$$

You have also learnt that a power of a product such as $(xy)^2$ can be expressed as a product of powers as x^2y^2 and a power of a quotient such as $\left(\frac{x}{y}\right)^2$ can be expressed as a quotient of powers as $\frac{x^2}{y^2}$.

Do the following review exercise to recall what you have learnt in previous grades regarding indices.

Review Exercise

1. Evaluate the following.

i. 2^5 **ii.** $(-3)^2$ **iii.** $(-4)^2$

iv. $\left(\frac{2}{3}\right)^2$ **v.** $(-3)^3$ **vi.** $(-4)^3$

2. Fill in the blanks.

i. $(xy)^2 = (xy) \times ......$
$= ...... \times ...... \times x \times y$
$= x \times x \times ...... \times ......$
$= x^2 \times y^2$

ii. $(pq)^3 = ... \times ...... \times ......$
$= p \times q \times ...... \times ......$
$= p \times p \times p \times ..... \times ..... \times .....$
$= p^3 \times q^3$

iii. $(2ab)^2 = ... \times ......$
$= ...... \times ...... \times b \times ...... \times ...... \times b$
$= 2 \times 2 \times ...... \times ...... \times ...... \times ......$
$= 4a^2b^2$

iv. $9p^2q^2 = ...^2 \times p^2 \times q^2$
$= .... \times .... \times p \times p \times .... \times ....$
$= (3 \times p \times q) \times (..... \times ..... \times .....)$
$= (3pq)^2$

3. Expand and write each of the following expressions as a product.

i. $2a^2$ **ii.** $3x^2y^2$ **iii.** $-5p^2q$

iv. $(-3)^5$ **v.** $(ab)^3$ **vi.** $x^4 \times y^4$

12.1 Products of powers with the same base

2^3 and 2^5 are two powers with the same base.
They can be expanded and written as follows.

$2^3 = 2 \times 2 \times 2$ and
$2^5 = 2 \times 2 \times 2 \times 2 \times 2$

Let us obtain the product of these two powers.

$$2^3 \times 2^5 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2)$$
$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$
$$= 2^8$$

2 is repeatedly multiplied three times in 2^3. Furthermore, 2 is repeatedly multiplied five times in 2^5.

Therefore, when multiplying 2^3 by 2^5, 2 is repeatedly multiplied 3 + 5 = 8 times.

This can be expressed as $2^3 \times 2^5 = 2^{3+5} = 2^8$.

It is important to remember that when two powers are multiplied, their indices can be added only if they have the same base. Then the base of the power that is obtained is also this common base.

Let us obtain the product $x^3 \times x^5$ accordingly. Since x^3 and x^5 have the same base, the two indices are added to get the product.

That is, $x^3 \times x^5 = x^{3+5}$
$= x^8$

This can be expressed as a law of indices as follows.

$$a^m \times a^n = a^{m+n}$$

This law can be extended to any number of powers.
For example,

$$a^m \times a^n \times a^p = a^{m+n+p}$$

Let us understand how this law is used to simplify expressions by considering some examples.

Example 1

Simplify the following.

i. $x^2 \times x^5 \times x$ ii. $a^2 \times b^2 \times a^2 \times b^3$ iii. $2x^2 \times 3x^5$

i.

$$x^2 \times x^5 \times x = x^{2+5+1} \text{ (since } x = x^1)$$
$$= \underline{\underline{x^8}}$$

ii.

$$a^2 \times b^2 \times a^2 \times b^3 = a^2 \times a^2 \times b^2 \times b^3$$
$$= a^{2+2} \times b^{2+3}$$
$$= a^4 \times b^5$$
$$= \underline{\underline{a^4 b^5}}$$

iii.

$$2x^2 \times 3x^5 = 2 \times x^2 \times 3 \times x^5$$
$$= 2 \times 3 \times x^2 \times x^5$$
$$= 6x^{2+5}$$
$$= \underline{\underline{6x^7}}$$

Do the following exercise using the law of indices on the product of powers.

Exercise 12.1

1. Fill in the blanks.

i. $2^5 \times 2^2$

$2^5 \times 2^2 = 2^{.... + ...}$
$= 2^{....}$

ii. $x^4 \times x^2$

$x^4 \times x^2 = x^{.... + ...}$
$= x^{....}$

iii. $a^3 \times a^4 \times a$

$a^3 \times a^4 \times a = a^{... + ... + ...}$
$= a^{....}$

iv. $5p^3 \times 3p$
$= 5 \times .... \times 3 \times ....$
$= 15p^{.... + ...}$
$= 15....$

v. $x^2 \times y^3 \times x^5 \times y^5$
$= x^{....} \times x^{....} \times y^{....} \times y^{....}$
$= x^{.... + ...} \times y^{.... + ...}$
$= ....^{....} \times ....^{....}$

2. Join each expression in column A with the expression in column B which is equal to it.

A	B
$x^3 \times x^7$	x^7
$x^5 \times x^2 \times x$	x^8
$x^7 \times x$	x^9
$x^2 \times x^2 \times x^6$	x^{10}
$x^2 \times x^3 \times x^2 \times x$	

3. Simplify and find the value.

i. $3^5 \times 3^5$ **ii.** $7^2 \times 7^3 \times 7$

4. Simplify.

i. $x^3 \times x^6$
ii. $x^2 \times x^2 \times x^2$
iii. $a^3 \times a^2 \times a^4$
iv. $2x^3 \times x^5$
v. $5p^2 \times 2p^3$
vi. $4x^2 \times 2x \times 3x^5$
vii. $m^2 \times 2n^2 \times m \times n$
viii. $2a^2 \times 3b^2 \times 5a \times 2b^3$

5. A pair of positive integral values that m and n can take so that the equation $x^m \times x^n = x^8$ holds true is 3 and 5. Write all such pairs of positive integral values.

6. Write a value of a for which the equation $a^2 + a^3 = a^5$ holds true and a value of a for which it is false.

12.2 Quotients of powers with the same base

Let us see whether there is a law of indices for the quotients of powers with the same base, similar to the one obtained above for the product.

$x^5 \div x^2$ can also be expressed as $\dfrac{x^5}{x^2}$.

Now, $\dfrac{x^5}{x^2} = \dfrac{x \times x \times x \times x \times x}{x \times x}$

$= x \times x \times x$

$= \underline{\underline{x^3}}$

$\therefore \dfrac{x^5}{x^2} = x^3$. When the index of the power in the numerator is 5 and the index of the power in the denominator is 2, then the index of the quotient is 5 – 2 = 3. The base of the quotient is x, which is the common base of the original two powers.

Therefore, $x^5 \div x^2$ can be simplified easily by subtracting the indices as follows.

$x^5 \div x^2 = x^{5-2} = x^3$

When powers with the same base are divided, the index of the divisor is subtracted from the index of the dividend. The base remains the same.

$$a^m \div a^n = a^{m-n}$$

It is important to remember this law of indices too.
Let us understand how this law is used to simplify expressions by considering the following examples.

Example 1

Simplify the following expressions.

a. $x^5 \times x^2 \div x^3$

$(x^5 \times x^2) \div x^3 = x^{5+2} \div x^3$

$= x^{7-3}$

$= \underline{\underline{x^4}}$

b. $4x^8 \div 2x^2$

$4x^8 \div 2x^2 = \dfrac{\overset{2}{\cancel{4}}x^8}{\underset{1}{\cancel{2}}x^2}$

$= 2x^{8-2}$

$= \underline{\underline{2x^6}}$

c. $\dfrac{a^3 \times a^2}{a}$

$\dfrac{a^3 \times a^2}{a} = a^{3+2-1}$

$= \underline{\underline{a^4}}$

Now do the following exercise.

Exercise 12.2

1. Simplify using the laws of indices.

i. $a^5 \div a^3$ **ii.** $\dfrac{x^7}{x^2}$ **iii.** $2x^8 \div x^3$ **iv.** $4p^6 \div 2p^3$

v. $\dfrac{10m^5}{2m^2}$ **vi.** $\dfrac{x^2 \times x^4}{x^3}$ **vii.** $n^5 \div (n^2 \times n)$ **viii.** $\dfrac{2x^3 \times 2x}{4x}$

ix. $\dfrac{x^5 \times x^2 \times 2x^6}{x^7 \times x^2}$ **x.** $\dfrac{a^5 \times b^3}{a^2 \times b^2}$ **xi.** $\dfrac{2p^4 \times 2q^3}{p \times q}$

2. Write five pairs of positive integral values for m and n which satisfy the equation $a^m \div a^n = a^8$

3. For each of the algebraic expressions in column A, select the algebraic expression in column B which is equal to it and combine the two expressions using the " = " sign.

A	B
$2a^5 \div 2a^2$	a
$a^6 \div a^4$	a^2
$\dfrac{a^7 \times a^2}{a^6}$	a^3
$\dfrac{a^3}{a}$	
$\dfrac{4a^5 \times a}{4a^3}$	

12.3 Negative indices

In the previous section we identified that $x^5 \div x^2$ is x^3. We know that this can be obtained by expanding $\dfrac{x^5}{x^2}$ and simplifying it as follows.

$$\frac{\cancel{x}^1 \times \cancel{x}^1 \times x \times x \times x}{\cancel{x}_1 \times \cancel{x}_1} = x^3$$

Let us simplify $x^2 \div x^5$ in a similar manner.

i. When expanded,

$$\frac{x^2}{x^5} = \frac{\overset{1}{\cancel{x}} \times \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \times \underset{1}{\cancel{x}} \times x \times x \times x}$$

$$= \underline{\underline{\frac{1}{x^3}}}$$

ii. Using laws of indices,

$$\frac{x^2}{x^5} = x^{2-5}$$

$$= \underline{\underline{x^{-3}}}$$

The two simplifications of $x^2 \div x^5$ obtained in (i) and (ii) above must be equal. Therefore, $\frac{1}{x^3} = x^{-3}$. Observe here that the index of the power in the denominator changes signs when it is brought to the numerator. This is an important feature related to indices which can be used when we need to change a negative index into a positive index. We can similarly write $x^3 = \frac{1}{x^{-3}}$.

This law can be expressed as follows.

$$x^n = \frac{1}{x^{-n}}$$

Accordingly, $a^{-m} = \frac{1}{a^m}$, $a^m = \frac{1}{a^{-m}}$, $\frac{a^{-m}}{a^{-n}} = \frac{a^n}{a^m}$ (By applying the above feature to both powers simultaneously)

We can use this law of indices to simplify algebraic expressions as shown in the following examples.

Example 1

Evaluate the following.

(i) 2^{-5} (ii) $\frac{1}{5^{-2}}$

i. $2^{-5} = \frac{1}{2^5}$

$$= \frac{1}{2 \times 2 \times 2 \times 2 \times 2}$$

$$= \underline{\underline{\frac{1}{32}}}$$

ii. $\frac{1}{5^{-2}} = 5^2$

$$= \underline{\underline{25}}$$

Example 2

Simplify: $\dfrac{2x^{-2} \times 2x^3}{2x^{-4}}$

$$\frac{2x^{-2} \times 2x^3}{2x^{-4}} = \frac{2 \times x^{-2} \times 2 \times x^3}{2 \times x^{-4}}$$

$$= \frac{\overset{1}{\cancel{2}} \times x^4 \times 2 \times x^3}{\underset{1}{\cancel{2}} \times x^2} \quad \left(\text{since } x^{-2} = \frac{1}{x^2} \text{ and } \frac{1}{x^{-4}} = x^4\right)$$

$$= \frac{2x^7}{x^2}$$

$$= 2x^{7-2}$$

$$= \underline{\underline{2x^5}}$$

Exercise 12.3

1. Write each of the following with positive indices.

i. 3^{-4} **ii.** x^{-5} **iii.** $2x^{-1}$ **iv.** $5a^{-2}$ **v.** $5p^2q^{-2}$

vi. $\dfrac{1}{x^{-5}}$ **vii.** $\dfrac{3}{a^{-2}}$ **viii.** $\dfrac{2x}{x^{-4}}$ **ix.** $\dfrac{a}{2b^{-3}}$ **x.** $\dfrac{m}{(2n)^{-2}}$

xi. $\dfrac{t^{-2}}{m}$ **xii.** $\dfrac{p}{q^{-2}}$ **xiii.** $\dfrac{x^{-2}}{2y^{-2}}$ **xiv.** $\left(\dfrac{2x}{3y}\right)^{-2}$

2. Evaluate the following.

i. 2^{-2} **ii.** $\dfrac{1}{4^{-2}}$ **iii.** 2^{-7} **iv.** $(-4)^{-3}$ **v.** 3^{-2}

vi. $\dfrac{5}{5^{-2}}$ **vii.** 10^{-3} **viii.** $\dfrac{3^{-2}}{4^{-2}}$

3. Simplify and write the answers with positive indices.

i. $a^{-2} \times a^{-3}$ **ii.** $a^2 \times a^{-3}$ **iii.** $\dfrac{a^2}{a^{-5}} \times a^{-8}$ **iv.** $2a^{-4} \times 3a^2$ **v.** $3x^{-2} \times 4x^{-2}$

vi. $\dfrac{10x^{-5}}{5x^2}$ **vii.** $\dfrac{4x^{-3} \times x^{-5}}{2x^2}$ **viii.** $\dfrac{(2p)^{-2} \times (2p)^3}{(2p)^4}$

12.4 Zero index

A power of which the index is zero, is known as a power with zero index. 2^0 is an example of a power with zero index.

When $x^5 \div x^5$ is simplified using the laws of indices we obtain,

$$x^5 \div x^5 = x^{5-5} = x^0$$

When it is expanded and simplified we obtain, $x^5 \div x^5 = \frac{x \times x \times x \times x \times x}{x \times x \times x \times x \times x}$

$= 1$

Since the answers obtained when $x^5 \div x^5$ is simplified by the two methods should be the same, we obtain $x^0 = 1$.

$x^0 = 1$ where, x is any number except 0.

This result is also used when simplifying algebraic expressions.

Note:

Example 1

Simplify.

i. $\frac{x^0 \times x^7}{x^2}$

$\frac{x^0 \times x^7}{x^2} = 1 \times x^7 \div x^2$

$= 1 \times x^{7-2}$

$= \underline{\underline{x^5}}$

ii. $\left(\frac{x^5 \times x^2}{a}\right)^0$

$\left(\frac{x^5 \times x^2}{a}\right)^0 = \underline{\underline{1}}$

(Since the whole term within brackets is the base, and 0 is the index, its value is 1.)

Let us improve our skills in simplifying expressions which contain powers with zero index by doing the following exercise.

Exercise 12.4

1. Simplify the following expressions.

i. $x^8 \div x^8$ **ii.** $(2p)^4 \times (2p)^{-4}$ **iii.** $\frac{a^2 \times a^3}{a \times a^4}$

iv. $\frac{y^4 \times y^2}{y^6}$ **v.** $\frac{p^3 \times p^5 \times p}{p^6 \times p^3}$ **vi.** $\frac{x^{-2} \times x^{-4} \times x^6}{y^{-2} \times y^8 \times y^{-6}}$

2. Evaluate the following.

i. $2^0 \times 3$ **ii.** $(-4)^0$ **iii.** $\left(\frac{x}{y}\right)^0 + 1$ **iv.** $\left(\frac{x^2}{y^2}\right)^0$ **v.** $5^0 + 1$

vi. $\left(\frac{2}{3}\right)^0$ **vii.** $(2ab)^0 - 2^0$ **viii.** $(abc)^0$

12.5 Power of a power

$(x^2)^3$ is the third power of x^2. A power of this type is known as a power of a power. We can simplify this as follows.

$$(x^2)^3 = x^2 \times x^2 \times x^2$$
$$(x^2)^3 = (x \times x) \times (x \times x) \times (x \times x)$$
$$= x \times x \times x \times x \times x \times x$$
$$= x^6$$

Therefore, $(x^2)^3 = x^6$.
Observe that the index 6 is 3 twos; that is, 2×3. Therefore, we can write $(x^2)^3 = x^{2\times 3} = x^6$.
Hence, when simplifying an algebraic expression of a power of a power, the indices are multiplied.
This is also a law of indices which can be expressed as follows.

$$(a^m)^n = a^{m\times n} = a^{mn}$$

Example 1

Simplify the following.

i. $(a^5)^2 \times a$ **ii.** $(p^3)^4 \times (x^2)^0$ **iii.** $(2x^2y^3)^2$

i. $(a^5)^2 \times a = a^{5\times 2} \times a$
$= a^{10} \times a^1$
$= a^{10+1}$
$= a^{11}$

ii. $(p^3)^4 \times (x^2)^0 = p^{3\times 4} \times x^{2\times 0}$
$= p^{12} \times x^0$
$= p^{12} \times 1$
$= p^{12}$

iii. $(2x^2y^3)^2 = (2 \times x^2 \times y^3)^2$
$= 2^2 \times x^4 \times y^6$
$= 4x^4y^6$

Let us improve our skills in simplifying expressions which contain the power of a power by doing the following exercise.

Exercise 12.5

1. Evaluate the following.

i. $(2^4)^2$ **ii.** $(3^2)^{-1}$ **iii.** $(2^3)^2 + 2^0$

iv. $(5^2)^{-1} + \frac{1}{5}$ **v.** $(4^0)^2 \times 1$ **vi.** $(10^2)^2$

2. Simplify the express using positive indices.

i. $(x^3)^4$ **ii.** $(p^{-2})^2$ **iii.** $(a^2 b^2)^2$ **iv.** $(2x^2)^3$

v. $\left(\frac{x^5}{x^2}\right)^3$ **vi.** $\left(\frac{a^3}{b^2}\right)^2$ **vii.** $\left(\frac{m^3}{n^2}\right)^{-2}$ **viii.** $(p^{-2})^{-4}$

ix. $(a^0)^2 \times a$

Miscellaneous Exercise

1. Evaluate the following.

i. $5^3 \times 5^2$ **ii.** $5^3 \div 5^2$ **iii.** $5^0 \times 5 \times 5^2$

iv. $(5^{-1})^2$ **v.** $\{(5^2)^0\}^4$ **vi.** $\frac{5^3 \times 5^{-1}}{(5^2)^2}$

vii. $5^2 \div 10^2$ **viii.** $5^2 \times 10^3 \times 5^{-1} \times 10^{-2}$

2. Simplify the following.

i. $(2x^5)^2$ **ii.** $(2ab^2)^3$ **iii.** $2x \times (3x^2)^2$ **iv.** $\frac{(4p^2)^3}{(2p^2q)^2}$

v. $\frac{(2p^2)^3}{3pq}$ **vi.** $\frac{(2a^2)^2}{5b^3} \times \frac{(3b^2)^2}{2a}$

Summary

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $x^n = \frac{1}{x^{-n}}$
- $(a^m)^n = a^{m \times n} = a^{mn}$
- $x^0 = 1$ where $x \neq 0$.