7 Axioms

By studying this lesson, you will be able to;

- identify five fundamental axioms of mathematics,
- develop geometrical relationships and solve problems involving calculations using the five fundamental axioms.

Axioms

Statements which are considered to be self-evident and are accepted without proof are called axioms. In mathematics, axioms are used to explain facts logically, develop relationships and reach conclusions.

Euclid, who is considered to be the father of geometry lived in Greece around 300 B.C. He introduced certain axioms related to mathematics in his book "Elements". Some of them are unique to geometry. Others are common axioms which can be used in other areas including algebra.

We consider five common axioms in this lesson. They can be summarized as given below.

1. Quantities which are equal to the same quantity, are equal.
2. Quantities which are obtained by adding equal quantities to equal quantities, are equal.
3. Quantities which are obtained by subtracting equal quantities from equal quantities, are equal.
4. Products which are equal quantities multiplied by equal quantities, are equal.
5. Quotients which are equal quantities divided by nonzero equal quantities, are equal.

By "quantities" we usually mean lengths, areas, volumes, masses, speeds, magnitudes of angles, etc.

These five axioms are very important because we can derive many results related to algebra and geometry by using them. Let us study these axioms in detail.

Axiom 1

> Quantities which are equal to the same quantity, are equal.

We can write this axiom briefly as given below.

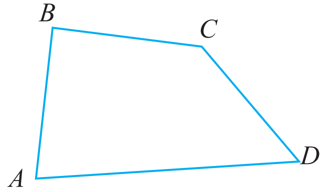
If $b = a$ and $c = a$, then $b = c$.

According to this axiom,

'If Hasith's age is the same as Kasun's and Harsha's age is also the same as Kasun's, then Hasith's age is the same as Harsha's.'

How Axiom 1 is used to obtain geometrical results is seen in the simple example given below.

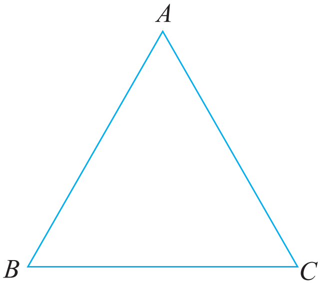
In the quadrilateral $ABCD$ shown below $BC = AB$ and $CD = AB$.



According to the above axiom,
$BC = CD$.

Example 1

In the triangle ABC, $AB = AC$ and $AB = BC$ If $AC = 5$ cm then determine the perimeter of the triangle ABC.



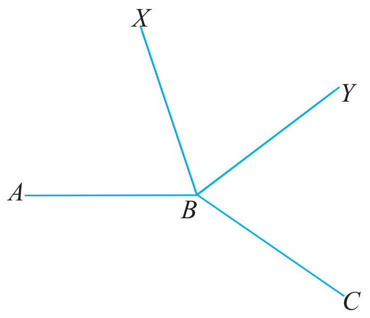
Since AC = 5 cm and $AC = AB$, according to Axiom 1, AB = 5 cm.
Since AB = 5 cm and $AB = BC$, according to Axiom 1, BC = 5 cm.
The perimeter of the triangle $ABC = AC + BC + AB$

= 5 cm + 5 cm + 5 cm

= 15 cm

Example 2

In the figure given below, $X\hat{B}Y = A\hat{B}X$ and $X\hat{B}Y = C\hat{B}Y$. Find the relationship between $A\hat{B}X$ and $C\hat{B}Y$.

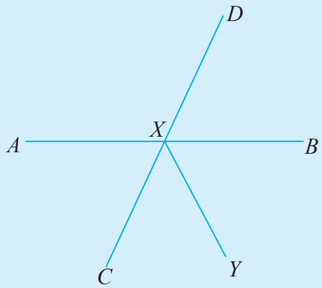


$$X\hat{B}Y = A\hat{B}X \text{ (given)}$$
$$X\hat{B}Y = C\hat{B}Y \text{ (given)}$$

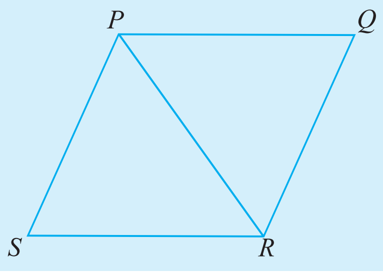
$\therefore$ According to Axiom 1, $A\hat{B}X = C\hat{B}Y$

Exercise 7.1

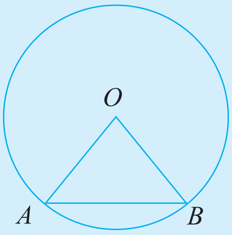
1. The straight lines AB and CD intersect at X. In the figure, $D\hat{X}B = B\hat{X}Y$. If $A\hat{X}C = 70^{\circ}$, find the magnitude of $B\hat{X}Y$.



2. In the parallelogram $PQRS$, $PQ = PR$ and $PQ = PS$. Based on its sides, mention what type of triangle PSR is.



3. The points A and B are located on the circle with centre O, such that $OA = AB$. Based on its sides, mention what type of triangle ABO is.



Axiom 2

Quantities which are obtained by adding equal quantities to equal quantities, are equal.

We can write this briefly as given below.

If $a = b$, then $a + c = b + c$.

This axiom can be written as given below too.

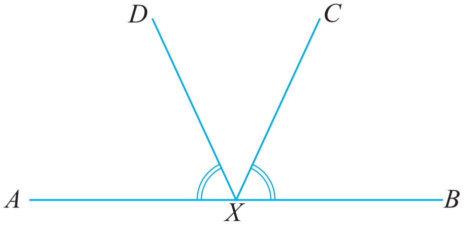
If $x = y$ and $p = q$, then $x + p = y + q$.

According to this axiom,

"If the cost incurred in purchasing vegetables is equal to the cost incurred in purchasing milk and the cost incurred in purchasing fruits is equal to the cost incurred in purchasing eggs, then the cost incurred in purchasing vegetables and fruits is equal to the cost incurred in purchasing milk and eggs."

Let us consider a simple geometrical result that can be derived using the above axiom.

In the figure given below, the point X is located on the straight line AB. Also, $A\hat{X}D = B\hat{X}C$.



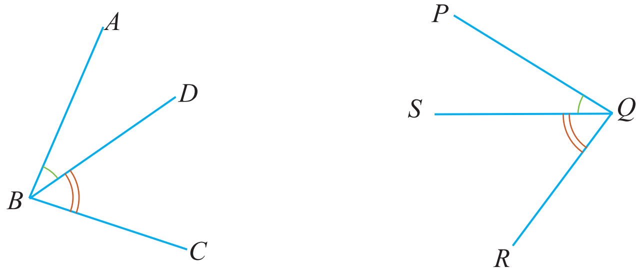
$$A\hat{X}D = B\hat{X}C \text{ (given)}$$

$\therefore$ According to Axiom 2, $\underbrace{A\hat{X}D + C\hat{X}D} = \underbrace{B\hat{X}C + C\hat{X}D}$

$$\therefore \; A\hat{X}C = B\hat{X}D$$

Example 1

In the figures given below, $A\hat{B}D = P\hat{Q}S$ and $C\hat{B}D = R\hat{Q}S$. Show that $A\hat{B}C = P\hat{Q}R$.



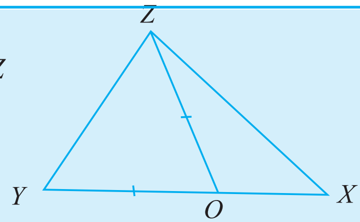
$A\hat{B}D = P\hat{Q}S$, $C\hat{B}D = R\hat{Q}S$.

$\therefore$ According to this axiom, $A\hat{B}D + C\hat{B}D = P\hat{Q}S + R\hat{Q}S$

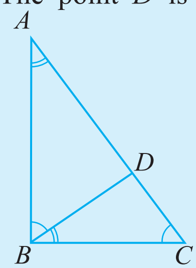
$\therefore A\hat{B}C = P\hat{Q}R$

Exercise 7.2

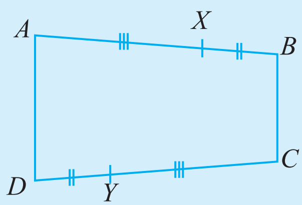
1. The point O is located on the side XY of the triangle XYZ such that $OZ = OY$. Show that $XY = OZ + OX$.



2. The point D is located on the side AC of the triangle ABC. If $A\hat{B}D = B\hat{C}D$ and $C\hat{B}D = B\hat{A}D$, show that $B\hat{A}D + B\hat{C}D = A\hat{B}C$.



3. The points X and Y are located on the sides AB and CD respectively of the quadrilateral $ABCD$, such that $AX = CY$ and $BX = DY$. Show that $AB = CD$.



Axiom 3

Quantities which are obtained by subtracting equal quantities from equal quantities, are equal.

We can write this briefly as given below.

If $a = b$, then $a - c = b - c$.

This axiom can be written as given below too.

If $a = b$ and $c = d$, then $a - c = b - d$.

A simple result in geometry that can be obtained by using the above axiom is given below.

In the figure given below, $AD = CB$.

$$AD = CB$$

According to Axiom 3, $AD - CD = CB - CD$

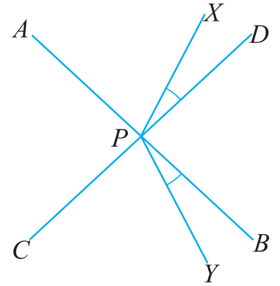
$$\therefore AC = DB$$

Example 1

The straight line segments AB and CD intersect at P. $X\hat{P}D = B\hat{P}Y$

(i) Show that $A\hat{P}X = C\hat{P}Y$.

(ii) If $A\hat{P}D = 95^\circ$ and $X\hat{P}D = 20^\circ$, find the magnitude of $C\hat{P}Y$.



(i) $A\hat{P}D = B\hat{P}C$ (vertically opposite angles)

$X\hat{P}D = B\hat{P}Y$ (given)

$\therefore$ According to this axiom, $\underbrace{A\hat{P}D - X\hat{P}D} = \underbrace{B\hat{P}C - B\hat{P}Y}$

$$\therefore A\hat{P}X = C\hat{P}Y$$

(ii) $A\hat{P}X = A\hat{P}D - X\hat{P}D$

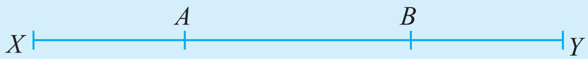
$A\hat{P}X = 95^{o} - 20^{o}$

$A\hat{P}X = 75^{o}$

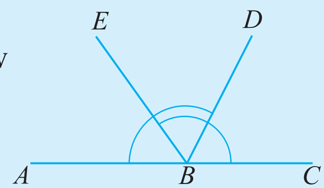
$\therefore\ C\hat{P}Y = 75^{o}$

Exercise 7.3

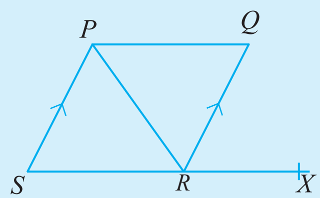
1. The points A and B are located on the line XY such that $XB = AY$. If $XY = 16$ cm and $BY = 6$ cm, find the length of AB.



2. The point B is located on the line AC. If $A\hat{B}D = C\hat{B}E$, show that $A\hat{B}E = C\hat{B}D$.



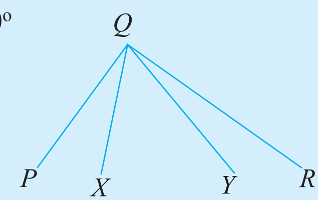
3. In the quadrilateral $PQRS$ in the figure, $S\hat{P}R = P\hat{R}Q$. If $Q\hat{P}S = P\hat{R}X$ and $S\hat{P}R = Q\hat{R}X$, show that $Q\hat{P}R = Q\hat{R}X$.



4. In the figure given here, $P\hat{Q}Y = X\hat{Q}R$. If $P\hat{Q}R = 110^{o}$ and $P\hat{Q}X = 35^{o}$,

(i) find the magnitude of $R\hat{Q}Y$.

(ii) find the magnitude of $X\hat{Q}Y$.



Axiom 4

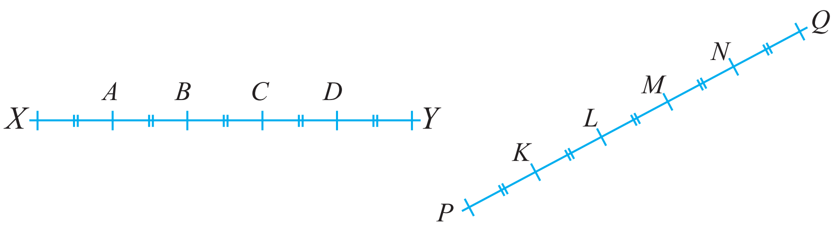
Products which are equal quantities multiplied by equals, are equal.

We can write this briefly as given below.

If $a = b$, then $ca = cb$.

Let us first consider an application of this axiom in geometry.

- As indicated in the figure given below, the points A, B, C and D are located on the straight line XY, such that $XA = AB = BC = CD = DY$. The points K, L, M and N are located on the straight line PQ, such that $PK = KL = LM = MN = NQ$. It is also given that $XA = PK$.



Let us show that $XY = PQ$.

Since $XA = AB = BC = CD = DY$, it is clear that $5\,XA = XY$.
Similarly, since $PK = KL = LM = MN = NQ$, we obtain that $5\,PK = PQ$.
Since $XA = PK$, by applying Axiom 4 we obtain $5\,XA = 5\,PK$.
$\therefore\; XY = PQ$.

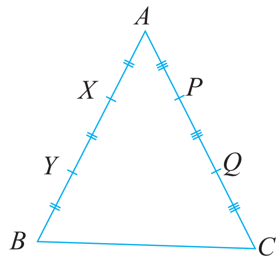
Although it is important to understand how results are derived by using axioms, in practice, when deriving geometrical results, the relevant axioms are not mentioned. This is because, as implied by the word "axiom", the validity of the derivation is obvious.

Now, let us consider how this axiom is applied in algebra.

If $x = 5$ and $y = 2x$, let us find the value of y.
Since $x = 5$, applying the above axiom and multiplying both sides by 2 we obtain $2x = 2 \times 5$.
But $2 \times 5 = 10$.
Hence by Axiom 1 we obtain
$y = 10$.

Example 1

The points X and Y are located on the side AB of the triangle ABC such that $AX = XY = YB$. The points P and Q are located on the side AC such that $AP = PQ = QC$. If $AX = AP$, find the relationship between AB and AC.



$$AX = XY = YB \quad \text{(given)}$$
$$\therefore AB = 3AX$$
$$AP = PQ = QC \quad \text{(given)}$$
$$\therefore AC = 3AP$$
$$AX = AP \text{ (given)}$$

According to Axiom 4;

$$3AX = 3AP$$
$$\therefore AB = AC$$

Axiom 5

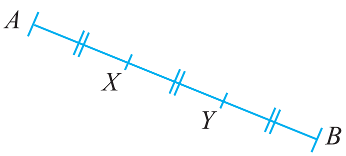
Quotients which are equal quantities divided by nonzero equals, are equal.

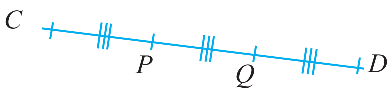
We can write this briefly as given below.

If $a = b$ then $\dfrac{a}{c} = \dfrac{b}{c}$.

Here, c is a nonzero number since it is meaningless to divide by zero.

- The line segments AB and CD in the figure are equal in length (that is, $AB = CD$). The points X and Y are located on AB such that $AX = XY = YB$. The points P and Q are located on CD such that $CP = PQ = QD$.





Let us show that $AX = CP$.

Since $AX = XY = YB$, we obtain that $\dfrac{AB}{3} = AX$.

Since $CP = PQ = QD$, we obtain that $\dfrac{CD}{3} = CP$.

Since $AB = CD$, according to Axiom 5,

$$\frac{AB}{3} = \frac{CD}{3}$$

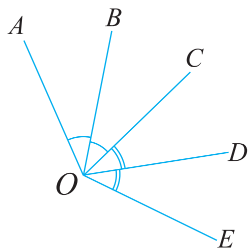
$\therefore AX = CP.$

Example 1

In the figure given below, $A\hat{O}B = B\hat{O}C$ and $C\hat{O}D = D\hat{O}E$. If $A\hat{O}C = C\hat{O}E$,

(i) find the relationship between $A\hat{O}B$ and $D\hat{O}E$.

(ii) If $B\hat{O}C = 35^{o}$, find the magnitude of $D\hat{O}E$.



(i) $A\hat{O}B = B\hat{O}C$ (given)

$\therefore A\hat{O}B = \dfrac{A\hat{O}C}{2}$

$C\hat{O}D = D\hat{O}E$ (given)

$\therefore D\hat{O}E = \dfrac{C\hat{O}E}{2}$

$A\hat{O}C = C\hat{O}E$ (given)

$\therefore$ According to Axiom 5, $\dfrac{A\hat{O}C}{2} = \dfrac{C\hat{O}E}{2}$

$\therefore \quad A\hat{O}B = D\hat{O}E$

(ii) $A\hat{O}B = B\hat{O}C$ (given) $\qquad B\hat{O}C = 35^{o}$ (given)

$\therefore A\hat{O}B = 35^{o}$ (by axiom 1)

$A\hat{O}B = D\hat{O}E$ (Proved above)

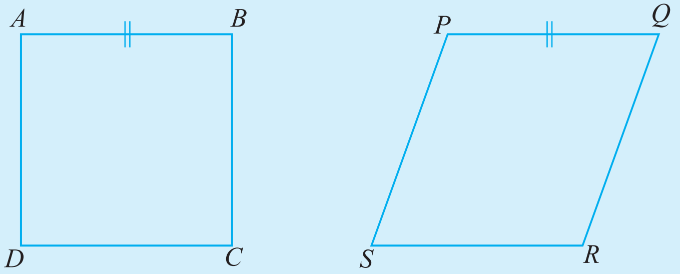
$\therefore D\hat{O}E = 35^{o}$ (by axiom 1)

Exercise 7.4

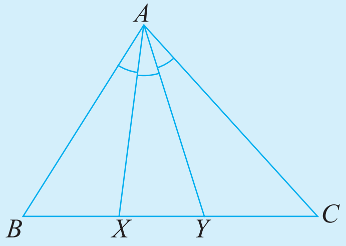
1. The square $ABCD$ and the rhombus $PQRS$ are such that $AB = PQ$.

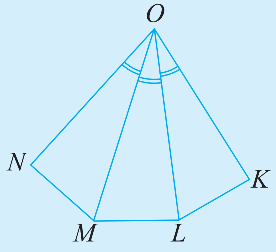
(i) Using Axiom 4, show that the perimeter of the square $ABCD$ is equal to the perimeter of the rhombus $PQRS$.

(ii) If AB = 7cm, find the perimeter of the rhombus $PQRS$.



2.





In the triangle ABC, $B\hat{A}X = X\hat{A}Y = C\hat{A}Y$. In the pentagon $KLMNO$ $M\hat{O}N = L\hat{O}M = K\hat{O}L$. If $B\hat{A}C = K\hat{O}N$,

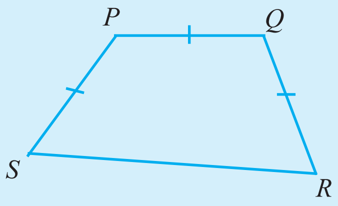
(i) show that $X\hat{A}Y = M\hat{O}L$.

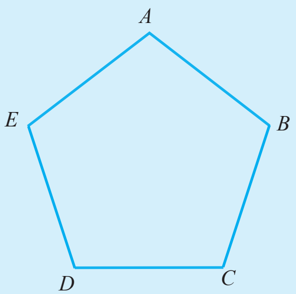
(ii) If $X\hat{A}Y = 30^{\circ}$, determine the magnitude of $K\hat{O}N$.

3. In the quadrilateral $PQRS$, $PQ = QR = SP$ and $2PQ = RS$. The perimeter of the regular pentagon $ABCDE$ is equal to that of the quadrilateral $PQRS$.

(i) Find the relationship between PQ and AB.

(ii) If AB = 8 cm, find the perimeter of the quadrilateral $PQRS$.





Further applications of the axioms

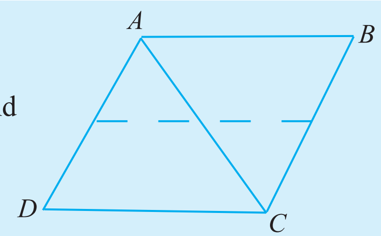
Example 1

Solve the equation given below using the axioms.
$2x + 5 = 13$

Here, solving the equation means finding the value of x. Since the quantity $2x + 5$ is equal to the quantity 13, according to Axiom 3, the quantities obtained by subtracting 5 from these two quantities are also equal.
$\therefore\ 2x + 5 - 5 = 13 - 5$.
Simplifying this we obtain,
$2x = 8$.
Since the quantity $2x$ is equal to the quantity 8, the quantities that are obtained by dividing these two quantities by 2 are also equal. Therefore, $\frac{2x}{2} = \frac{8}{2}$.

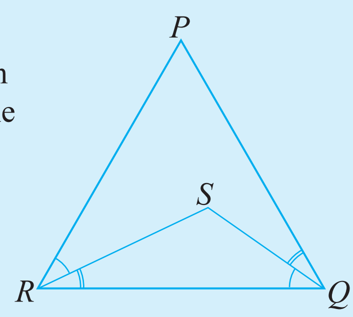
When we simplify this we obtain $x = 4$. That is, the solution of the equation is 4.

Miscellaneous Exercise



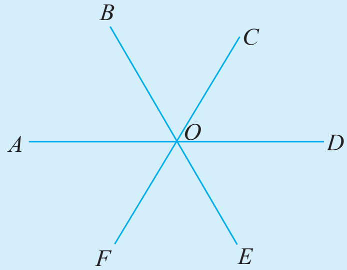
1. In the quadrilateral $ABCD$, $AD = AC$, $BC = AC$, $AB = BC$ and $AD = CD$. Show that $ABCD$ is a rhombus.

2. As indicated in the figure, the point S is located such that $P\hat{R}S = S\hat{Q}R$ and $Q\hat{R}S = P\hat{Q}S$. By applying the axioms,



(i) show that $P\hat{R}Q = P\hat{Q}R$,
(ii) show that if $R\hat{P}Q = P\hat{R}Q$, then all the angles of the triangle PQR are equal in magnitude.

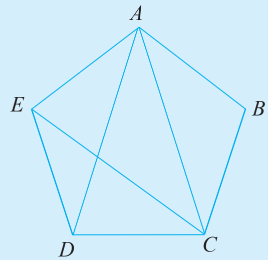
3. As indicated in the figure, the straight lines AD, BE and CF intersect at the point O. If $D\hat{O}E = A\hat{O}F$, show that $B\hat{O}D = D\hat{O}F$.



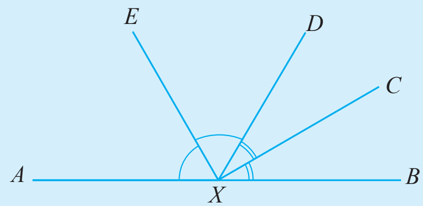
4. In the regular pentagon $ABCDE$, $E\hat{A}D = D\hat{A}C = B\hat{A}C$ and $B\hat{C}A = A\hat{C}E = D\hat{C}E$.

(i) Show that $B\hat{C}A = B\hat{A}C$.

(ii) If $B\hat{A}C = 36^{\circ}$ find the magnitude of $C\hat{D}E$.



5. The point X lies on the straight line AB. Also, $A\hat{X}E = E\hat{X}D$ and $B\hat{X}C = C\hat{X}D$. Determine the magnitude of $C\hat{X}E$.



Summary

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- Quantities which are equal to the same quantity, are equal.
- Quantities which are obtained by adding equal quantities to equal quantities, are equal.
- Quantities which are obtained by subtracting equal quantities from equal quantities, are equal.
- Products which are equal quantities multiplied by equal quantities, are equal.
- Quotients which are equal quantities divided by nonzero equal quantities, are equal.