6

Factors of Algebraic Expressions

By studying this lesson, you will be able to;

- factorize algebraic expressions with four terms when the factors are binomial expressions,
- factorize trinomial quadratic expressions of the form $x^2 + bx + c$,
- factorize algebraic expressions written as a difference of two squares.

Factors of algebraic expressions

The meanings of many algebraic terms were explained in the previous lesson. In this lesson we will consider what is meant by the factors of an algebraic expression (or an algebraic term).

Consider the term $2xy$. It is formed by the product of 2, x and y. Therefore, 2, x and y are all factors of $2xy$.

$2x + 2y$ is a binomial expression. It is the sum of two algebraic terms. 2 and x are factors of $2x$. Similarly, 2 and y are factors of $2y$. Accordingly, 2 is a factor of both the terms $2x$ and $2y$. You have learnt in grade 8 that the above binomial expression can be written as $2\,(x + y)$ by factoring out the common factor 2. Hence;

$$2x + 2y = 2\,(x + y)$$

What is important here is that the algebraic expression $2x + 2y$, which is the sum of $2x$ and $2y$, is expressed as a product of 2 and $x + y$. We say that 2 and $x + y$ are factors of $2x + 2y$. That is, the algebraic expression $2x + 2y$ can be expressed as a product of its factors 2 and $x + y$.

One factor of the above algebraic expression $2x + 2y$ is the number 2 and another factor is the algebraic expression $x + y$. However, an algebraic expression could also be expressed as a product of algebraic terms or algebraic expressions. For example, since the expression $xy + 5xz$ can be written as $x(y + 5z)$, x and $y + 5z$ are factors of it.

According to the facts learnt in lesson 5, when the algebraic expression $x(y + 5z)$ which is a product is expanded, we obtain the algebraic expression $xy + 5xz$, which is a sum of algebraic terms. In this lesson we will study the inverse of the process that was learnt in lesson 5. That is, we will learn how to write a given algebraic expression as a product of factors.

Observe how each algebraic expression given below has been written as a product of factors as learnt in grade 8.

- $3x + 12 = 3\ (x + 4)$
- $6a + 12b - 18 = 6\ (a + 2b - 3)$
- $-2x - 6y = -2\ (x + 3y)$
- $3x - 6xy = 3x\ (1 - 2y)$

In the second example above, the common factor of the terms of the expression $6a + 12b - 18$ is 6. Observe that this is the highest common factor of 6, 12 and 18. When a number is a common factor, we should always consider the highest common factor. Furthermore, when factorizing algebraic expressions, the numbers need not be factorized further. For example, $6x + 6y$ is written as $6(x + y)$ and not as $2 \times 3\ (x + y)$.

Do the following review exercise to establish these facts further.

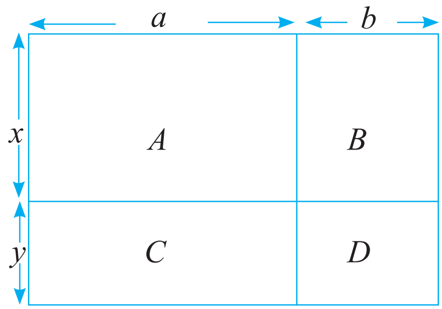
Review Exercise

Write each of the following algebraic expressions as a product of factors.

a. $8x + 12y$	**b.** $9a + 18y$	**c.** $3m + 6$
d. $20a - 30b$	**e.** $4p - 20q$	**f.** $12 - 4k$
g. $3a + 15b - 12$	**h.** $12a - 8b + 4$	**i.** $9 - 3b - 6c$
j. $-12x + 4y$	**k.** $-8a - 4b$	**l.** $-6 + 3m$
m. $ab + ac$	**n.** $p - pq$	**o.** $ab + ac - ad$
p. $3x + 6xy$	**q.** $6ab - 9bc$	**r.** $4ap + 4bp - 4p$
s. $x^3 + 2x$	**t.** $3m - 2nm^2$	**u.** $6s - 12s^2t$

6.1 Factors of algebraic expressions with four terms

The figure of a large rectangle which is composed of the four rectangular sections A, B, C and D is given below.



Let us find the area of each rectangle in terms of the given algebraic symbols x, y, a and b.

The area of section $A = a \times x = ax$

The area of section $B = b \times x = bx$

The area of section $C = a \times y = ay$

The area of section $D = b \times y = by$

Now let us find the area of the large rectangle.

The length of the large rectangle = $a + b$
The breadth of the large rectangle = $x + y$
Hence, the area of the large rectangle = $(a + b)\,(x + y)$

Now, since the total area of the 4 small rectangles = the area of the large rectangle, $ax + ay + bx + by = (a + b)\,(x + y)$.

We can verify the total above equality by expanding the product $(a + b)(x + y)$ by using the method learnt in the previous lesson.

Let us expand it as follows.

$$\begin{aligned}(a + b)(x + y) &= a\,(x + y) + b\,(x + y)\\ &= ax + ay + bx + by\end{aligned}$$

The validity of the equality is verified.

In this lesson we expect to learn how to write an expression of the form $ax + bx + ay + by$ as a product of two factors as $(a + b)\,(x + y)$. First, we need to observe that the four terms ax, ay, bx and by have no common factors. Therefore, the factoring out of a common factor cannot be done directly. However, if we consider the four terms pairwise, the expression can be factored as follows.

$$ax + bx + ay + by = (ax + bx) + (ay + by)$$
$$= x(a + b) + y(a + b)$$

The final expression is the sum of the two expressions $x(a + b)$ and $y(a + b)$. Now observe that the two expressions $x(a + b)$ and $y(a + b)$ have a common factor $(a + b)$. Therefore, by factoring out this expression, we can rewrite the given expression as a product of two factors as $(a + b)(x + y)$.

$$ax + bx + ay + by = x(a + b) + y(a + b)$$
$$= (a + b)(x + y)$$

Example 1

Factorize $3x + 6y + kx + 2ky$.

$$3x + 6y + kx + 2ky = 3(x + 2y) + k(x + 2y)$$
$$= \underline{\underline{(x + 2y)(3 + k)}}$$

Example 2

Factorize $a^2 - 3a + ab - 3b$.

$$a^2 - 3a + ab - 3b = a(a - 3) + b(a - 3)$$
$$= \underline{\underline{(a - 3)(a + b)}}$$

Example 3

Factorize $x^2 + xy - x - y$.

$$x^2 + xy - x - y = x^2 + xy - 1(x + y)$$
$$= x(x + y) - 1(x + y)$$
$$= \underline{\underline{(x + y)(x - 1)}}$$

Exercise 6.1

Factorize each of the following algebraic expressions.

a. $ax + ay + 3x + 3y$
b. $ax - 8a + 3x - 24$
c. $mp - mq - np + nq$
d. $ak + al - bk - bl$
e. $x^2 + 4x - 3x - 12$
f. $y^2 - 7y - 2y + 14$
g. $a^2 - 8a + 2a - 16$
h. $b^2 + 5b - 2b - 10$
i. $5 + 5x - y - xy$
j. $ax - a - x + 1$

6.2 Factors of trinomial quadratic expressions of the form of $x^2 + bx + c$

Recall how we obtained the product of the two algebraic expressions $(x + 3)$ and $(x + 4)$.

$$\begin{aligned}(x + 3)(x + 4) &= x(x + 4) + 3(x + 4)\\ &= x^2 + 4x + 3x + 12\\ &= x^2 + 7x + 12\end{aligned}$$

Since we have obtained $x^2 + 7x + 12$ as the product of $(x + 3)$ and $(x + 4)$, the expressions $(x + 3)$ and $(x + 4)$ are factors of $x^2 + 7x + 12$. Expressions of the form $x^2 + 7x + 12$ consisting of three terms of which one is a quadratic term are called trinomial quadratic expressions.

Note:
The trinomial quadratic expressions we consider here can in general be written in the form $x^2 + bx + c$. Here b and c are numerical values. For example, $x^2 + 7x + 12$ is the trinomial quadratic expression that is obtained when $b = 7$ and $c = 12$. In general, bx is called the middle term and c is called the constant term. The expression $x^2 + 7x + 12$ can be written as a product of two factors as $(x + 3)(x + 4)$. However, there are some trinomial quadratic expressions which cannot be written as a product of two such factors, as for example, the expression $x^2 + 3x + 4$.

Here, we only consider how to find the factors of those trinomial quadratic expressions that can be written as a product of two factors.

To find out how to write a trinomial quadratic expression as a product of two binomial terms, let us analyze the steps we carried out in obtaining the product of two binomial expressions, in the opposite direction.

- In the trinomial quadratic expression $x^2 + 7x + 12$, the middle term $7x$ has been written as a sum of two terms as $3x + 4x$.

  There are many ways of writing $7x$ as a sum of two terms. For example, $7x = 5x + 2x$ and $7x = 8x + (-x)$. The importance of $3x$ and $4x$ can be explained as follows.

- The product of $3x$ and $4x = 3x \times 4x = 12x^2$.
- Moreover, the product of the first and last terms of the trinomial quadratic expression $x^2 + 7x + 12$ is also $x^2 \times 12 = 12x^2$.

The observations from the above analysis can be used to factorize trinomial quadratic expressions. The middle term should be written as a sum of two terms. Their product should be equal to the product of the first and last terms of the expression to be factorized.

Let us factorize $x^2 + 6x + 8$. The middle term is $6x$. It should be written as a sum of two terms, and their product should be equal to $x^2 \times 8 = 8x^2$.

Based on the above facts, we have to find a pair of linear terms of which the product is $8x^2$ and the sum is $6x$. The table below shows the possible ways of writing $8x^2$ as a product of two linear terms.

Pair of linear terms	**Product**	**Sum**
x, $8x$	$x \times 8x = 8x^2$	$x + 8x = 9x$
$2x$, $4x$	$2x \times 4x = 8x^2$	$2x + 4x = 6x$

According to the table, it is clear that the middle term $6x$ is obtained from $2x + 4x$. Let us factorize the expression $x^2 + 6x + 8$.

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + 2x + 4x + 8 \\ &= x(x+2) + 4(x+2) \\ &= (x+2)(x+4) \end{aligned}$$

$\therefore$ $x + 2$ and $x + 4$ are factors of $x^2 + 6x + 8$.

Instead of writing the middle term as $2x + 4x$, let us write it as $4x + 2x$ and factorize to see whether we obtain different factors.

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + 4x + 2x + 8 \\ &= x(x+4) + 2(x+4) \\ &= (x+4)(x+2) \end{aligned}$$

The same pair of factors are obtained. Therefore, we see that the order in which the selected pair is written does not affect the final answer.

Example 1

Factorize $x^2 + 5x + 6$.
In this expression,
the product of the first and last terms $= x^2 \times 6 = 6x^2$
The middle term $= 5x$

We can factorize this expression as below, because $2x + 3x = 5x$ and $(2x)(3x) = 6x^2$

$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 2x + 3x + 6 \\ &= x(x+2) + 3(x+2) \\ &= \underline{\underline{(x+2)(x+3)}} \end{aligned}$$

Example 2

Factorize $x^2 - 8x + 12$.

The product of the first and last terms of the expression is $x^2 \times 12 = 12x^2$ and the middle term is $(-8x)$. Here we have a negative term. The table given below shows the various ways in which two terms in x can be selected such that their product is $12x^2$.

x,	$12x$
$2x$,	$6x$
$3x$,	$4x$
$-2x$,	$-6x$
$-3x$,	$-4x$
$-x$,	$-12x$

According to the table, if we write $-8x = (-2x) + (-6x)$, then we obtain $(-2x)(-6x) = 12x^2$.

$$\begin{aligned} \text{Hence, } x^2 - 8x + 12 &= x^2 - 2x - 6x + 12 \\ &= x(x-2) - 6(x-2) \\ &= \underline{\underline{(x-2)(x-6)}} \end{aligned}$$

Example 3

Factorize $y^2 + 2y - 15$.

The product of the first and last terms of the expression $= y^2 \times -15 = -15y^2$

The middle term $= 2y$

By writing $-15y^2 = (5y)(-3y)$, the middle term is obtained as $(5y) + (-3y) = 2y$

Therefore,

$$\begin{aligned} y^2 + 2y - 15 &= y^2 - 3y + 5y - 15 \\ &= y(y-3) + 5(y-3) \\ &= \underline{\underline{(y-3)(y+5)}} \end{aligned}$$

Example 4

Factorize $a^2 - a - 20$.

The product of the first and last terms of the expression is $= a^2 \times (-20) = -20a^2$ and the middle term is $(-a)$.

Since $-20a^2 = (-5a)(4a)$ and $(-5a) + (4a) = -a$, the expression can be factored as follows.

$$\begin{aligned} a^2 - a - 20 &= a^2 + 4a - 5a - 20 \\ &= a(a+4) - 5(a+4) \\ &= (a+4)(a-5) \end{aligned}$$

Exercise 6.2

Factorize the quadratic expressions given below.

a. $x^2 + 9x + 18$	**b.** $y^2 + 11y + 30$	**c.** $a^2 + 10a + 24$
d. $b^2 - 8b + 15$	**e.** $x^2 - 5x + 6$	**f.** $m^2 - 12m + 20$
g. $a^2 + a - 12$	**h.** $p^2 + 5p - 24$	**i.** $p^2 + 6p - 16$
j. $x^2 - x - 12$	**k.** $a^2 - 3a - 40$	**l.** $r^2 - 3r - 10$
m. $y^2 + 6y + 9$	**n.** $k^2 - 10k + 25$	**o.** $4 + 4x + x^2$
p. $36 + 15x + x^2$	**q.** $30 - 11a + a^2$	**r.** $54 - 15y + y^2$

Note:

When factorizing trinomial quadratic expressions, writing the middle term as a sum of two suitable terms is an important step. Although a specific method has been given above to find the two terms, an easier method is to write the middle term as a sum of two terms and check whether their product is equal to the product of the first and last terms of the given expression. This skill can be mastered with practice. However, once the two terms have been written, we have to be careful when simplifying the expression. In example 4 above, when the common factor -5 is factored out from the expression $-5a - 20$, we obtain $-5(a+4)$ This is often mistakenly written as $-5(a-4)$.

6.3 Factors of an expression written as a difference of two squares

Consider the product of the two binomial expressions $(x - y)$ and $(x + y)$.

$$\begin{aligned} (x-y)(x+y) &= x(x+y) - y(x+y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

Accordingly, $(x + y)(x - y)$ is equal to the expression $x^2 - y^2$. The expression $x^2 - y^2$ is said to be a difference of two squares.

The fact that $(x + y)(x - y) = x^2 - y^2$ means that $x + y$ and $x - y$ are factors of $x^2 - y^2$.

Let us see whether we can find the factors of $x^2 - y^2$ by considering it as a quadratic expression in x. We can rewrite it as a trinomial quadratic expression in x by writing the middle term as 0. We then obtain the expression $x^2 + 0 - y^2$. Now consider its factorization.

The product of the first and last terms of the expression is $= x^2 \times (-y^2) = -x^2y^2$ and the middle term is 0.

Now,

$$-x^2y^2 = (-xy) \times (xy) \text{ and } -xy + xy = 0$$

Therefore, $x^2 + 0 - y^2 = x^2 - xy + xy - y^2$

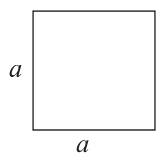
$= x(x - y) + y(x - y)$

$= (x - y)(x + y)$

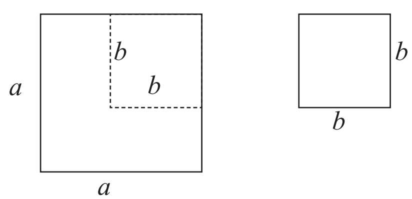
Again we obtain $x^2 - y^2 = (x - y)(x + y)$.

Now let us consider how to find the factors of the difference of two squares by considering areas.

Consider a square of side length a units.



From this square, cut out a square of side length b units.



The area of the remaining portion is $a^2 - b^2$ square units.

Let us rearrange the remaining portion as follows.

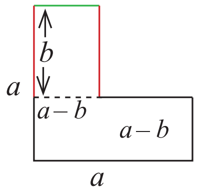


Figure I

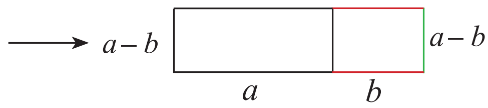


Figure II

The area of the remaining portion according to figure II is $(a-b)(a+b)$.
Accordingly, $a^2 - b^2 = (a-b)(a+b)$.

Now let us consider some examples of the factorization of expressions which are the difference of two squares.

Example 1

Factorize $x^2 - 25$.

$$x^2 - 25 = x^2 - 5^2$$
$$= \underline{\underline{(x-5)(x+5)}}$$

Example 2

Factorize $9 - y^2$.

$$9 - y^2 = 3^2 - y^2$$
$$= \underline{\underline{(3-y)(3+y)}}$$

Example 3

Factorize $4a^2 - 49$.

$$4a^2 - 49 = 2^2a^2 - 7^2$$
$$= \underline{\underline{(2a-7)(2a+7)}}$$

Example 4

Factorize $1 - 4b^2$.

$$1 - 4b^2 = 1^2 - 2^2b^2$$
$$= \underline{\underline{(1-2b)(1+2b)}}$$

Example 5

Factorize $2x^2 - 72$.

$$2x^2 - 72 = 2(x^2 - 36)$$
$$= 2(x^2 - 6^2)$$
$$= \underline{\underline{2(x-6)(x+6)}}$$

Example 6

Find the value $33^2 - 17^2$.

$$33^2 - 17^2 = (33 + 17)(33 - 17)$$
$$= 50 \times 16$$
$$= \underline{\underline{800}}$$

Example 7

Factorize $\frac{x^2}{4} - \frac{1}{9}$.

$$\frac{x^2}{4} - \frac{1}{9} = \frac{x^2}{2^2} - \frac{1}{3^2}$$

$$= \underline{\underline{\left(\frac{x}{2} + \frac{1}{3}\right)\left(\frac{x}{2} - \frac{1}{3}\right)}}$$

Example 8

Factorize $1 - \frac{9x^2}{16}$.

$$1 - \frac{9x^2}{16} = 1^2 - \left(\frac{3x}{4}\right)^2$$

$$= \underline{\underline{\left(1 - \frac{3x}{4}\right)\left(1 + \frac{3x}{4}\right)}}$$

Exercise 6.3

Factorize the expressions given below.

a. $x^2 - 100$	**b.** $m^2 - 36$	**c.** $p^2 - 81$
d. $4 - b^2$	**e.** $16 - a^2$	**f.** $64 - y^2$
g. $x^2 - 4y^2$	**h.** $9a^2 - 16b^2$	**i.** $100x^2 - 1$
j. $25m^2 - n^2$	**k.** $49 - 81p^2$	**l.** $25a^2b^2 - 9c^2$

Miscellaneous Exercise

1. Factorize the following algebraic expressions by changing the order in which the terms appear as required.

i. $ax + by - ay - bx$ **ii.** $9p - 2q - 6q + 3p$

iii. $x - 12 + x^2$ **iv.** $4 - k^2 - 3k$

2. Factorize the following algebraic expressions.

i. $8x^2 - 50$ **ii.** $3x^2 - 243$

iii. $a^3 b^3 - ab$ **iv.** $3 - 12q^2$

3. Find the value.

i. $23^2 - 3^2$ **ii.** $45^2 - 5^2$ **iii.** $102^2 - 2^2$

4. Join each algebraic expression in column A with the product of its factors in column B.

A	B
$x^2 - x - 6$	$\left(\frac{x}{5} - 1\right)\left(\frac{x}{5} + 1\right)$
$x^2 + 5x - 3x - 15$	$2x(x - 2)(x + 2)$
$2x^3 - 8x$	$(x - 3)(x + 5)$
$4x^2 - 9m^2$	$(x - 3)(x + 2)$
$\frac{x^2}{25} - 1$	$(2x - 3m)(2x + 3m)$