2 Binary Numbers

By studying this lesson, you will be able to;

- identify binary numbers,
- convert a decimal number into a binary number,
- convert a binary number into a decimal number,
- add and subtract binary numbers,
- identify instances where binary numbers are used.

Introduction

Let us recall how numbers are written in the Hindu - Arabic number system, which is the number system we use.

As an example, let us consider the number 3 725. According to what we have learnt in previous grades,

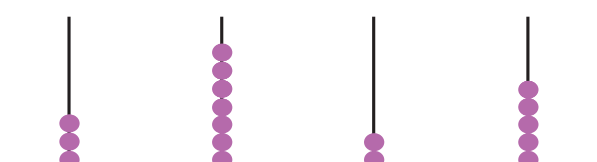
5 denotes the number of 1s (that is, the number of 10^0s),

2 denotes the number of 10s (that is, the number of 10^1s),

7 denotes the number of 100s (that is, the number of 10^2s),

3 denotes the number of 1000s (that is, the number of 10^3s).

The above can be represented on an abacus as shown below.



1000	100	10	1
10^3	10^2	10^1	10^0

Observe that the number 3 725 can also be written in terms of powers of 10 as shown below.

$$3\ 725 = 3 \times 1000 + 7 \times 100 + 2 \times 10 + 5 \times 1$$

That is, $3\ 725 = 3 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$

If we consider 603 as another example, we can write it as shown below.

$$603 = 6 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$$

In the Hindu - Arabic number system which we use, the place values are powers of ten such as 1, 10, 100 and 1000. Moreover, we use the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 to write numbers in the Hindu - Arabic number system. The method of writing numbers using these 10 digits and assigning place values which are powers of ten, is called writing the numbers in "base 10". When studying about number bases, these numbers are called "decimal numbers".

Note : $10^0 = 1$. Similarly, any nonzero base raised to the power zero is always equal to one. Accordingly, $2^0 = 1$.

2.1 Expressing numbers in the binary number system

We can use number bases other than base ten to express numbers. For example, we can express numbers in "base two" by using only the digits 0 and 1, and assigning place values which are powers of two. To do this, let us first identify several powers of two.

We can write them as;

$2^0 = 1$	$2^5 = 32$
$2^1 = 2$	$2^6 = 64$
$2^2 = 4$	$2^7 = 128$
$2^3 = 8$	$2^8 = 256$
$2^4 = 16$	$2^9 = 512$

To understand the method of writing numbers in base two, let us first consider the base ten number 13 as an example. Let us see how we can write 13 as a sum of powers of two.

The first few powers of two are;

1, 2, 4 and 8.

Using these numbers which are powers of two, we can write,

$$13 = 8 + 4 + 1$$

i.e., $13 = 2^3 + 2^2 + 2^0$

i.e., $13 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$.

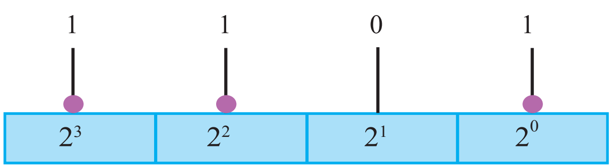
Here, we have written all the non - negative powers of two in descending order, starting from 2^3 and continuing with 2^2, 2^1 and 2^0. Also, since the power 2^3 is included, it is written as 1×2^3 and since the power 2^1 is not included, it is written as 0×2^1. Recall that we use only the digits 1 and 0 when writing base two numbers. Considering the above facts, we can write 13 as a base two number as follows.

1101

The digits 0 and 1 appearing in this base two number can be described as follows.

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

We can also express it using an abacus as follows.



To indicate that 1101 is a base two number, we usually write "two" as a subscript and express the number as 1101_{two}. In this lesson, whenever necessary, we indicate base ten numbers with the subscript "ten" to differentiate the base two numbers from the base ten numbers. For example, the decimal number 603 is written as 603_{ten}.

Let us consider another example. Let us write the base ten number 20_{ten} as a base two number.

By recalling the powers of two, we can write;

$$\begin{aligned} 20 &= 16 + 4 \\ &= 2^4 + 2^2 \\ &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0. \end{aligned}$$

Hence,

$20_{ten} = 10100_{two}$.

There is an important fact to remember here. There is only one way of writing any number as a sum of distinct descending powers of two. For example, $20 = 16 + 4$ can only be written as $2^4 + 2^2$ as a sum of distinct descending powers of two. There is no other way. You can see this for yourself by attempting to find a different way. Moreover, any number can be written as a sum of powers of two. You can verify this too by writing different decimal numbers as a sum of distinct powers of two.

The above method of writing a decimal number as a sum of distinct descending powers of two, cannot be considered as a precise method. The reason for this is because it is difficult to decide what powers of two add up to a number, when the number is large. For example, it is not easy to determine the powers of two that add up to the decimal number 3905_{ten}.

Therefore, let us now consider another method that can be used to convert any decimal number to a binary number fairly easily.

Consider 22_{ten} as an example. To write this as a binary number, we need to first divide 22 by 2 and write the remainder also.

$$\begin{array}{r|l} 2 & \underline{22} \\ & 11 \quad \text{remainder } 0 \end{array}$$

Next we need to divide the quotient 11 which we obtained by dividing 22 by 2, again by 2.

$$\begin{array}{r|ll} 2 & \underline{22} & \\ 2 & \underline{11} & \text{remainder } 0 \\ & \;\;5 & \text{remainder } 1 \end{array}$$

We need to continue dividing the quotient by 2 and noting down the remainder, until we get 0 as the quotient and 1 as the remainder. The complete division is shown below.

$$\begin{array}{r|rll} 2 & \underline{22} & & \\ 2 & \underline{11} & \text{remainder} & \boxed{0} \\ 2 & \underline{5} & \text{remainder} & \boxed{1} \\ 2 & \underline{2} & \text{remainder} & \boxed{1} \\ 2 & \underline{1} & \text{remainder} & \boxed{0} \\ & 0 & \text{remainder} & \boxed{1} \end{array}$$

$$1\;0\;1\;1\;0_{\text{two}}$$

When the highlighted remainders are written from bottom to top, we obtain the required base two number.

$$22_{\text{ten}} = 10110_{\text{two}}$$

Let us see whether we can verify this answer using the method we discussed earlier of writing the number as a sum of powers of two.

$$22 = 16 + 4 + 2 = 2^4 + 2^2 + 2^1 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$
$$= 10110_{\text{two}}$$

The answer is verified.

Example 1

Write each decimal number given below as a binary number.

i. 32_{ten}

$$\begin{array}{r|r l}
2 & 32 & \\
\hline
2 & 16 & 0 \\
\hline
2 & 8 & 0 \\
\hline
2 & 4 & 0 \\
\hline
2 & 2 & 0 \\
\hline
2 & 1 & 0 \\
\hline
 & 0 & 1
\end{array}$$

$32_{ten} = 100000_{two}$

ii. 154_{ten}

$$\begin{array}{r|r l}
2 & 154 & \\
\hline
2 & 77 & 0 \\
\hline
2 & 38 & 1 \\
\hline
2 & 19 & 0 \\
\hline
2 & 9 & 1 \\
\hline
2 & 4 & 1 \\
\hline
2 & 2 & 0 \\
\hline
2 & 1 & 0 \\
\hline
 & 0 & 1
\end{array}$$

$154_{ten} = 10011010_{two}$

Exercise 2.1

Convert the decimal numbers (base ten numbers) given below into binary numbers (base two numbers).

a. 4 **b.** 9 **c.** 16 **d.** 20 **e.** 29

f. 35 **g.** 43 **h.** 52 **i.** 97 **j.** 168

2.2 Converting binary numbers into decimal numbers

Decimal numbers were converted into binary numbers in the above section 2.1. In this section we consider the inverse process; that is, converting binary numbers into decimal numbers. This can be done fairly easily. Let us learn how to do this by considering an example.

In section 2.1, when we wrote the decimal number 13 as a binary number, we obtained 1101_{two}. Let us recall what the digits 1, 1, 0 and 1 represent.

1 1 0 1

number of 2^3s — number of 2^2s — number of 2^1s — number of 2^0s

Therefore, by adding all the values of the powers of two in 1101_{two} we get the corresponding decimal representation.

$$\begin{aligned} 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 &= 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 \\ &= 8 + 4 + 1 = 13. \end{aligned}$$

By simplifying, we obtain the corresponding decimal number 13.

Example 1

Write 101100_{two} as a decimal number.

First, it should be noted that the place value of the leftmost digit of this base two number is 2^5 and that the other place values are obtained by reducing the index by one (starting from 5) for each move from left to right. Then the required decimal number can be found by multiplying each power of two (place value) by the relevant co-efficient and adding all the terms together.

$$101100_{two} = 1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 0\times 2^0$$
$$= 2^5 + 2^3 + 2^2 = 32 + 8 + 4$$
$$= 44_{ten}$$

Therefore, when 101100_{two} is written in base 10 we obtain 44_{ten}.

Note: This answer can be verified by converting 44_{ten} back into a binary number.

Exercise 2.2

Convert the binary numbers given below into base ten numbers (decimal numbers).

a. 101_{two} **b.** 1101_{two} **c.** 1011_{two} **d.** 1100_{two} **e.** 11111_{two}

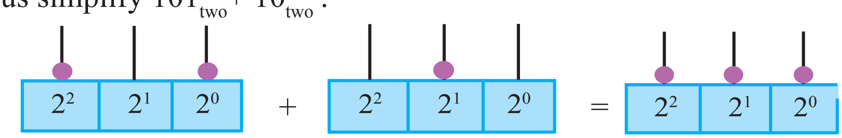
f. 100111_{two} **g.** 110111_{two} **h.** 111000_{two} **i.** 111110_{two} **j.** 110001_{two}

2.3 Adding binary numbers

When representing binary numbers on an abacus, the maximum number of counters that can be placed on a rod is 1. Moreover, instead of placing two counters on a rod, we always place one counter on the rod to the left of it.

Let us learn how to add binary numbers with the aid of two abacuses.

Let us simplify $101_{two} + 10_{two}$.



101_{two} 10_{two} 111_{two}

A B C

Let us represent the sum of the numbers represented on the abacuses A and B on the abacus C.
When we consider the two abacuses A and B;

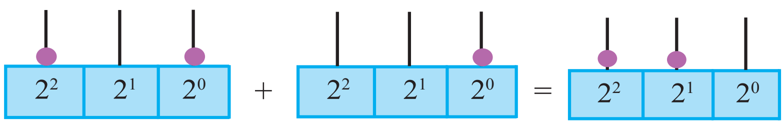
the sum of the counters on the 2^0 rods is 1,

the sum of the counters on the 2^1 rods is 1,

the sum of the counters on the 2^2 rods is 1.

Therefore, $101_{two} + 10_{two} = 111_{two}$

Now, let us obtain the value of $101_{two} + 1_{two}$ using the abacuses.



A B C

The counter on the 2^0 rod in A and the counter on the 2^0 rod in B, cannot both be placed on the 2^0 rod in C, because there cannot be two counters on any rod of an abacus used to represent a binary number. Instead, one counter should be placed on the rod to the left of the 2^0 rod. This is shown on the rod 2^1 in C.

Therefore, $101_{two} + 1_{two} = 110_{two}$.

This is clarified further by adding the numbers vertically.

$$\begin{array}{r} 101_{two} \\ +\quad 1_{two} \\ \hline 110_{two} \\ \hline\hline \end{array}$$

Adding from right to left; first, one 2^0s + one 2^0s = one 2^1s and zero 2^0s.

Then, one 2^1s + zero 2^1s = one 2^1s. Finally, one 2^2s + zero 2^2s = one 2^2s.

Example 1

Find the value.

(i) $11101_{two} + 1101_{two}$ (ii) $1110_{two} + 111_{two}$

(i)
$$\begin{array}{r} {\color{red}{11\;\;1}} \\ 11101_{two} \\ +\ \ 1101_{two} \\ \hline 101010_{two} \\ \hline\hline \end{array}$$

(ii)
$$\begin{array}{r} {\color{red}{11}} \\ 1110_{two} \\ +\ \ 111_{two} \\ \hline 10101_{two} \\ \hline\hline \end{array}$$

Note: When adding binary numbers observe the relationships given below.

$$1_{two} + 0_{two} = 1_{two}$$
$$1_{two} + 1_{two} = 10_{two}$$
$$1_{two} + 1_{two} + 1_{two} = 11_{two}$$

Exercise 2.3

1. Find the value.

a. $\begin{array}{r} 111_{two} \\ +101_{two} \\ \hline \end{array}$

b. $\begin{array}{r} 10111_{two} \\ +\ \ 1011_{two} \\ \hline \end{array}$

c. $\begin{array}{r} 1011_{two} \\ +\ 11101_{two} \\ \hline \end{array}$

d. $11101_{two} + 1110_{two}$

e. $11011_{two} + 11_{two}$

f. $100111_{two} + 11_{two} + 1_{two}$

g. $11_{two} + 111_{two} + 1111_{two}$

h. $11110_{two} + 1110_{two} + 110_{two}$

2. Fill each cage with the suitable digit.

a. $\begin{array}{r} 11_{two} \\ +1\square_{two} \\ \hline 1\square 1_{two} \\ \hline\hline \end{array}$

b. $\begin{array}{r} 1\ 10\ \square_{two} \\ +\ \square 1\ 1_{two} \\ \hline 1\square 100_{two} \\ \hline\hline \end{array}$

c. $\begin{array}{r} 10\ 0\ 1_{two} \\ +\ \ \square\ 1\ \square_{two} \\ \hline \square 00\square 0_{two} \\ \hline\hline \end{array}$

d. $\begin{array}{r} 1\ 1\ 1\ 0_{two} \\ +\ \ \ 1\ \square\ \square_{two} \\ \hline 10\square\ 0\ 1_{two} \\ \hline\hline \end{array}$

e. $\begin{array}{r} 1\ \square\ 1\ \square_{two} \\ +\ \ \ 1\ \square\ 1_{two} \\ \hline 1\square\ 0\,0\,0_{two} \\ \hline\hline \end{array}$

f. $\begin{array}{r} 1\ 1\ \square\ 1_{two} \\ +1\ 1\ 1\ 0_{two} \\ \hline 1\square\ \square\ 1\ \square_{two} \\ \hline\hline \end{array}$

2.4 Subtracting binary numbers

When adding binary numbers, we saw that whenever we obtained a sum of 2 in a particular position, we replaced it with 1 in the position left of it.

$$\begin{array}{r} 101_{two} \\ +\quad 1_{two} \\ \hline 110_{two} \end{array} \qquad \text{(right hand column: } 1_{two} + 1_{two} = 10_{two}\text{)}$$

Now let us find the value of $110_{two} - 1_{two}$. According to the above addition, the answer should be 101_{two} . Let us consider how this answer is obtained.

$$\begin{array}{rccc} & 2^2 & 2^1 & 2^0 \\ & 1 & 1 & 0_{two} \\ - & & & 1_{two} \\ \hline & 1 & 0 & 1_{two} \end{array}$$

We cannot subtract 1 from 0 in the rightmost column. Therefore, let us take 1 from the 2^1 column which is to the left of it.The value of this is 2 in the 2^0 column. Now we subtract 1 from this 2 to obtain 1 in the rightmost column. Now there is 0 instead of 1 in the column 2^1.

Therefore, $110_{two} - 1_{two} = 101_{two}$.

Example 1

$$\begin{array}{r} 1101_{two} \\ -\quad 111_{two} \\ \hline 110_{two} \end{array}$$

Let us check the accuracy of the answer by considering $110_{two} + 111_{two}$.

$$110_{two} + 111_{two} = \underline{\underline{1101}}_{\ two}$$

Note: It is very important to develop the habit of checking the accuracy of an answer to a subtraction problem using addition as shown above.

Exercise 2.4

1. Find the value.

a. $\begin{array}{r} 11_{two} \\ -\ 1_{two} \\ \hline \end{array}$

b. $\begin{array}{r} 10_{two} \\ -\ 1_{two} \\ \hline \end{array}$

c. $\begin{array}{r} 101_{two} \\ -\ 1_{two} \\ \hline \end{array}$

d. $\begin{array}{r} 101_{two} \\ -\ 11_{two} \\ \hline \end{array}$

e. $111_{two} - 11_{two}$

f. $110_{two} - 11_{two}$

g. $1100_{two} - 111_{two}$

h. $\begin{array}{r} 10001_{two} \\ -\ 111_{two} \\ \hline \end{array}$

i. $\begin{array}{r} 100000_{two} \\ -\ 11011_{two} \\ \hline \end{array}$

j. $\begin{array}{r} 100011_{two} \\ -\ 10001_{two} \\ \hline \end{array}$

k. $11000_{two} - 1111_{two}$

l. $101010_{two} - 10101_{two}$

2.5 Applications of binary numbers

The fundamental digits in the binary number system are 0 and 1. Many modern digital instruments are made based on this feature. When designing lighting system circuits, "current off" and "current on" conditions are represented by 0 and 1.

If ⊛ is used to represent the current "on" condition and ○ to represent the current "off" condition, then the combination ⊛ ○ ○ ⊛ is represented by 1001_{two}. The storing of data and computations done in computers and calculators are based on this concept. Any number system can be developed under the same principles used to develop the binary number system. Storing of data can be done using other number systems too.

Note: If a number system is developed using base four, only the fundamental digits 0, 1, 2 and 3 are used.
For example, the decimal number 4 is expressed as 10_{four} in this number system.
In the base five number system, the fundamental digits are 0, 1, 2, 3 and 4, and the decimal number 5 is expressed as 10_{five} in this system.

Miscellaneous Exercise

1. Find the value.

a. $1101_{two} + 111_{two} - 1011_{two}$

b. $11111_{two} - (101_{two} + 11_{two})$

c. $110011_{two} - 1100_{two} - 110_{two}$

2. Write the next number, after adding 1 to each given number. 1_{two}, 11_{two}, 111_{two}, 1111_{two}, 11111_{two}, 111111_{two}

3. Represent the decimal number 4^2 as a binary number.

4. i. Simplify $49_{ten} - 32_{ten}$ and convert the answer into a binary number.
ii. Convert 49_{ten} and 32_{ten} into binary numbers and find their difference. See whether the answer you obtain is the same as the answer in (1) above.

Summary

Summary

- In the binary number system, the fundamental digits are 0 and 1.
- The place values of the binary number system are; 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, etc.