1 Number Patterns

By studying this lesson you will be able to;

- develop the general term of a number pattern with the same difference between adjacent terms,
- develop the number pattern when the general term is known,
- solve problems associated with number patterns.

Introduction to number patterns

Several number patterns are given below.

i. 3, 3, 3, 3, 3, ...
ii. 2, 4, 6, 8, 10, ...
iii. 5, 8, 11, 14, 17, ...
iv. 2, 4, 8, 16, 32, ...
v. 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ...
vi. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...

The first number pattern is very simple. Every number of this number pattern is 3. While the first number of the second number pattern is 2, all the numbers thereafter are obtained by adding 2 to the previous number.

While the first number of the third number pattern is 5, all the numbers thereafter are obtained by adding 3 to the previous number.

While the first number of the fourth number pattern is 2, all the numbers thereafter are obtained by multiplying the previous number by 2.

The fifth and sixth number patterns have characteristics which are inherent to them.

The numbers of a number pattern are called "terms".

For example, each term of the first number pattern is 3.

The first term of the second number pattern is 2, the second term is 4, the third term is 6, etc. In this pattern, each term which comes after the first term is obtained by adding 2 to the previous term.

The first term of the third number pattern is 5, the second term is 8, the third term is 11, etc. In this pattern, each term which comes after the first term is obtained by adding 3 to the previous term.

In the fourth number pattern, each term which comes after the first term is obtained by multiplying the previous term by 2.

The ways in which the terms of the fifth and sixth number patterns are obtained can also be described as above. However, the descriptions will be more complicated.

Observe that the terms of the number patterns given above are separated by commas and that there are three dots (ellipsis) at the end of each number pattern. This is how number patterns are usually written. The three dots indicate that the number pattern continues.

In mathematics, the word “sequence” is used for the word “pattern”.
Accordingly, six “number sequences” (or simply “sequences”) are given above.
The order of the terms of a sequence is important.

For example, although the sequence 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, ...
and the sequence 1, 2, 1, 2, 3, 4, 3, 4, 5, 6, 5, 6, ... consist of the same numbers, they are two different sequences.

In the above examples of sequences, only a few initial terms are given. However, it is incorrect to presume the pattern of the sequence by considering only a few initial terms.
For example,

1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, ...

is a number pattern; that is, a sequence. If only the first five terms of the sequence are given (that is, 1, 2, 3, 4, 5, …), and a person is asked what the next term is, one may be provided with the incorrect answer 6. Hence, asking for the next term (or next few terms) after giving only the first few terms of a sequence is mathematically incorrect.

A method of describing a sequence accurately is by providing a rule by which each term of the sequence can be calculated.

The uniqueness (or characteristic) of the second and third sequences of the six sequences given above can be explained as follows.

In the second sequence, every term which comes after the first term is obtained by adding the constant value 2 to the previous term. This can be illustrated as follows.

2 4 6 8 10

+2 +2 +2 +2

In the third sequence, every term which comes after the first term is obtained by adding the constant value 3 to the previous term. This can be illustrated as follows.

5 8 11 14 17

+3 +3 +3 +3

Here, the meaning of "constant value" is "the value remains unchanged".

The characteristic which is common to both these patterns can be described as follows.

"The value obtained by subtracting the previous term from any term (except the first term) is a constant (that is, a constant value)."

The value of this constant is 2 for the sequence 2, 4, 6, 8, 10, …
(since $4 - 2 = 6 - 4 = 8 - 6 = 10 - 8 = 2$).

The value of this constant is 3 for the sequence 5, 8, 11, 14, 17, …
(since $8 - 5 = 11 - 8 = 14 - 11 = 17 - 14 = 3$).

Let us study further about sequences of which the difference between every pair of consecutive terms is a constant value.

This constant value is known as the common difference of the sequence. Accordingly,

common difference = any term except the first term – the previous term

It can be seen that the first sequence 3, 3, 3, 3, 3, … also has the same characteristic.

3 3 3 3 3

+0 +0 +0 +0

Here, the constant value added (that is the common difference) is 0.

Another sequence with the same characteristic is given below.

17 12 7 2 − 3...

−5 −5 −5 −5

The first term of this sequence is 17. Every term which comes thereafter is obtained by subtracting 5 from the previous term. That is, by adding −5 to the previous term. Accordingly, the common difference of this sequence is −5.

Common difference = 12 − 17 = 7 − 12 = 2 − 7 = −3 − 2 = −5.

If the value of the common difference and the first term of a sequence with a common difference are known, the first few terms of the sequence can be written easily. A couple of examples of such sequences are given below.

Example 1

The first three terms of the sequence with first term 4 and common difference 3 are 4, 7 and 10.

Example 2

The first five terms of the sequence with first term 7 and common difference − 4 are 7, 3. −1, −5 and −9.

The first few terms of a sequence with a common difference can be written easily, when the first term and common difference are given. But it is not so easy to find, say the 50th term or the 834th term. The reason is because 50 and 834 are fairly large numbers.
It is important to know the general term of a sequence to be able to find any term of the sequence easily. Now let us see what is meant by the general term.

The general term of a number pattern

First, let us introduce a specific notation to denote the terms of a sequence. For a given sequence, let us denote

the first term by T_1,
the second term by T_2,
the third term by T_3, etc.

For example, with regard to the sequence

$$5, 11, 17, 23, \ldots$$

we can indicate the terms as follows:

the first term, $T_1 = 5$
the second term, $T_2 = 11$
the third term, $T_3 = 17$
the fourth term, $T_4 = 23$

It is very important to consider the *n*th term of a sequence, as is usually done in mathematics. Here, n represents any positive integer. The reason for this is that the values n can assume are positive integers such as 1, 2, 3, ... The $\frac{1}{2}$th term, the −4th term, the 3.5th term, etc., have no meaning. When considering a value n, the corresponding *n*th term is denoted by T_n. This is called the **general term** of the sequence. Accordingly,

the general term (*n*th term) of a sequence is denoted by T_n.

1.1 Developing the number sequence when the general term is given

In the previous section we learnt the notation that is used to denote the terms of a sequence, in particular, the general term. Now, through a couple of examples, let us consider how to develop the number sequence and how to find a named term of the sequence, when the general term is given.

Example 1

Consider the number sequence with general term $T_n = 2n + 3$.

(i) Write the first three terms of this sequence.
(ii) Find the 20th term.
(iii) Which term is equal to 123?
(iv) Find the $(n + 1)$th term in terms of n.

(i) Since the general term $T_n = 2n + 3$,
when $n = 1$; the first term $T_1 = (2 \times 1) + 3 = 2 + 3 = 5$,
when $n = 2$; the second term $T_2 = (2 \times 2) + 3 = 4 + 3 = 7$,
when $n = 3$; the third term $T_3 = (2 \times 3) + 3 = 6 + 3 = 9$.

Therefore, the first three terms of this number pattern are 5, 7, 9.

(ii) The 20th term is obtained by substituting $n = 20$ in $2n + 3$.

The 20th term, $T_{20} = (2 \times 20) + 3 = 40 + 3$
$= 43$

Therefore, the 20th term is 43.

(iii) Let us assume that the nth term is 123.

$$\begin{aligned} \text{Then, } 2n + 3 &= 123 \\ 2n + 3 - 3 &= 123 - 3 \\ 2n &= 120 \\ n &= \frac{120}{2} \\ &= 60 \end{aligned}$$

Therefore, 123 is the 60th term of the number pattern.

(iv) In order to obtain the $(n + 1)$th term, let us substitute $(n + 1)$ for n.
The $(n + 1)$th term,

$$\begin{aligned} T_{n+1} &= 2(n + 1) + 3 \\ &= 2n + 2 + 3 \\ &= 2n + 5 \end{aligned}$$

Therefore, the $(n + 1)$th term, in terms of n, is $2n + 5$.

Example 2

Consider the number pattern with general term $T_n = 56 - 4n$.

(i) Write the first three terms of this number pattern.
(ii) Find the 12th term.
(iii) Show that 0 is a term of this number pattern.
(iv) Show that 18 is not a term of this number pattern.

(i) Since the general term $T_n = 56 - 4n$,

when $n = 1$; the first term $T_1 = 56 - (4 \times 1) = 56 - 4 = 52$
when $n = 2$; the second term $T_2 = 56 - (4 \times 2) = 56 - 8 = 48$
when $n = 3$; the third term $T_3 = 56 - (4 \times 3) = 56 - 12 = 44$
Therefore, the first three terms of the number pattern are 52, 48, 44.

(ii)The 12th term $= 56 - 4 \times 12$
$= 56 - 48$
$= 8$

(iii) If 0 is a term of the number pattern, then for some integer n, we have

$$56 - 4n = 0.$$
$$\therefore\ 56 - 4n + 4n = 4n \text{ (adding } 4n \text{ to both sides)}$$
$$\frac{56}{4} = \frac{4n}{4}$$
$$14 = n$$
$$n = 14$$

The 14th term of the number pattern is 0.Therefore 0 is a term of this number pattern.

(iv) If 18 is a term of this number pattern, then for some integer n, we have

$$56 - 4n = 18.$$
Then, $56 - 4n + 4n = 18 + 4n$
$$56 - 18 = 18 - 18 + 4n$$
$$38 = 4n$$
$$9\frac{1}{2} = n$$

If 18 is a term of this number pattern, the value of n should be a whole number.
Since $n = 9\frac{1}{2}$, 18 is not a term of this number pattern.

Exercise 1.1

1. Complete the table.

The general term of the number pattern	The first term when $n = 1$	The second term when $n = 2$	The third term when $n = 3$	First three terms of the number pattern
$3n + 2$	$(3 \times 1) + 2 = 5$	$(3 \times 2) + 2 = 8$	$(3 \times 3) + 2 = 11$	5, 8, 11
$5n - 1$	$(5 \times 1) - 1 = 4$	....	....	..., ..., ...
$2n + 5$	....	....	....	..., ..., ...
$20 - 2n$	....	....	....	..., ..., ...
$50 - 4n$	....	....	....	..., ..., ...
$35 - n$	....	....	....	..., ..., ...

2. The general term of a number pattern is $4n - 3$.

i. Write the first three terms of this number pattern.
ii. Find the 12th term.
iii. Which term is equal to 97?
iv. Show that 75 is not a term of this number pattern.

3. Consider the number pattern with nth term $7n + 1$.

i. Write the first three terms of this number pattern.
ii. Find the 5th term.
iii. Which term is equal to 36?
iv. Write the $(n+1)$th term, in terms of n.

4. Consider the number pattern with general term $T_n = 50 - 7n$.

i. Write the first three terms of this number pattern.
ii. Find the 10th term.
iii. Write the $(n + 1)$th term, in terms of n.
iv. Show that the terms which come after the 7th term are negative numbers.

1.2 Obtaining an expression for the general term (T_n)

In the previous section an expression was given for the general term T_n. Our objective in this section, is to obtain an expression for T_n in terms of n. Then, any term of the sequence can easily be found by using the obtained expression. Now let us consider how we can develop such an expression, through an example.
Suppose we want to find the 80th term of the sequence 5, 11, 17, 23…, which is a sequence with a common difference. That is, we want to find the value of T_{80}. First, examine the pattern given in the following table.

n	T_n	How T_n can be written in terms of n and the common difference 6.
1	5	$6 \times 1 - 1$ or $5 + 0 \times 6$
2	11	$6 \times 2 - 1$ or $5 + 1 \times 6$
3	17	$6 \times 3 - 1$ or $5 + 2 \times 6$
4	23	$6 \times 4 - 1$ or $5 + 3 \times 6$
5	29	$6 \times 5 - 1$ or $5 + 4 \times 6$

You may be wondering why the expressions $6 \times 1 - 1$, $6 \times 2 - 1$, $6 \times 3 - 1$, etc., given in the 3rd column of the table have been written. Especially, the reason why 1 is subtracted from each term may be unclear to you. This can be explained as follows.

Since the common difference of the given sequence 5, 11, 17, 23, … is 6, let us write the given sequence first and several multiples of 6 below it.

5, 11, 17, 23, 29, ...

6, 12, 18, 24, 30, ...

It is clear that the given sequence can be obtained by subtracting 1 from each multiple of 6.

That is,

the first term of the sequence = the first multiple of 6 – 1
the second term of the sequence = the second multiple of 6 –1
the third term of the sequence = the third multiple of 6 – 1

Accordingly,

the nth term of the sequence = the nth multiple of 6 – 1

$$\therefore \; T_n = 6n - 1$$

Accordingly,

$$T_{80} = 6 \times 80 - 1 = 479.$$

Therefore, the 80th term is 479.

Moreover, an expression for the general term T_n of this sequence was found above as $T_n = 6n - 1$.
We can find any term of the sequence using this expression. For example, in order to find the 24th term of this sequence, n has to be substituted with 24.

$T_{24} = 6 \times 24 - 1 = 143$

Therefore, the 24th term of the sequence is 143.

Let us consider another example.

Example 1

Given that the sequence with first four terms15, 19, 23, 27 has a common difference, let us find an expression for the nth term.

The common difference $= 19 - 15 = 4$.
Let us write the first few terms of the given sequence, and several multiples (positive integer multiples) of 4 below them.

15, 19, 23, 27, ...
4, 8, 12, 16, ...

It is clear that the given number pattern is obtained by adding 11 to each multiple of 4.

Therefore, the expression for the general term T_n is given by $T_n = 4n + 11$.
Let us find the 100th term using this expression.

$T_{100} = 4 \times 100 + 11 = 411$

Now let us consider a sequence with a negative common difference, which therefore consists of terms which are decreasing in value.

Example 2

Let us find an expression for the general term T_n of the sequence with a common difference, of which the first four terms are 10, 7, 4, 1.

The common difference of the sequence 10, 7, 4, 1, … $= 7 - 10 = -3$.

Therefore, let us write the first few terms of the given sequence and a few multiples of –3 (integral), one below the other.

$$10, 7, 4, \ldots$$
$$-3, -6, -9, \ldots$$

It can be seen that the terms of the sequence are obtained by adding 13 to the multiples of −3. Therefore, the general term can be written as

$$T_n = -3n + 13$$

(Or else, it can be written as $T_n = 13 - 3n$ with the positive term first.)

For example, in order to find the 30th term, $n = 30$ should be substituted in the expression for T_n.

$T_{30} = -3 \times 30 + 13 = -77$

Therefore, the 30th term is − 77.

Exercise 1.2

All sequences in this exercise have a common difference.

1. Copy the following table in your exercise book and complete it.

Pattern	**The difference between two successive terms**	**The number, whose multiples are used to develop the pattern**
5, 8, 11, 14,...	$8 - 5 = 3$	3
10, 17, 24, 31, ...		
$2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$		
20, 17, 14, 11, ...		
50, 45, 40, 35, ...		
0.5, 0.8, 1.1, 1.4, ...		

2. Complete the table in relation to the number pattern 10, 17, 24, 31, ...,

Sequential order of the terms	**Term**	**How the pattern has been developed**
1st term	10	$7 \times 1 + ...$
2nd term	17	$7 \times 2 + ...$
3rd term	24	$... + ...$
4th term	31	$... + ...$
nth term	....	$... + ... = ...$

3. Find the general term of each of the number patterns given below.

a. 1, 4, 7, 10, ...
b. 1, 7, 13, 19, ...
c. 9, 17, 25, 33, ...
d. 4, 10, 16, 22, ...
e. 22, 19, 16, 13, ...
f. 22, 20, 18, 16, ...

1.3 Solving mathematical problems involving number patterns

We can solve various mathematical problems by developing number patterns using information that is given.

Example 1

A long distance runner trains every day. On the first day he runs 500 m and on each day thereafter he runs 100 m more than on the previous day.

i. Write separately the distances he runs on the first three days.
ii. Find the general term T_n for the distance he runs on the nth day, in terms of n.
iii. Find the distance he runs on the 20th day.
iv. On which day does he run 3km?

i. The distance run on the first day = 500 m
The distance run on the second day = 500 m + 100 m = 600 m
The distance run on the third day = 500 m + 100 m + 100 m = 700 m
$\therefore$ The first three terms of the number pattern are 500, 600, 700

ii. Let us take the day as n.

The number pattern of the distance run by the athlete is built up by multiples of 100.

Therefore, the general term $T_n = 100n + 400$

iii. It is clear that the distance run on the 20th day is represented by the 20th term.

$$\begin{aligned} \text{The 20th term, } T_{20} &= (100 \times 20) + 400 \\ &= 2000 + 400 \\ &= 2400 \text{ m} \\ &= 2.4 \text{ km} \end{aligned}$$

$\therefore$ The distance run on the 20th day is 2.4 km.

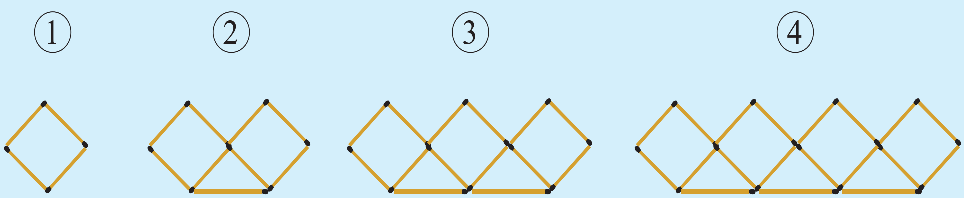
iv. Let us assume that 3000 m are run on the nth day.

$$\begin{aligned} \text{Then, } 100n + 400 &= 3000 \\ 100n + 400 - 400 &= 3000 - 400 \\ 100\,n &= 2600 \\ \therefore\ n &= \frac{2600}{100} \\ &= 26 \end{aligned}$$

Therefore, 3km are run on the 26th day.

Exercise 1.3

1. A pattern created by using matchsticks is shown below.



Complete the table in relation to the pattern given above.

Figure Number	1	2	3	4
Total number of matchsticks	....	9	....	....

i. Find the number of matchsticks needed to create the 20th figure.

ii. 219 matchsticks are required to create which figure of this pattern?

iii. Show that one matchstick remains after creating a figure of this pattern by using the maximum number of match sticks from 75 matchsticks.

2. A worker cuts pieces of rods of different lengths from iron rods which are 5 m in length, in order to build a gate by welding pieces together. The length of the shortest piece of iron rod that is cut is 15 cm. All the other pieces are cut such that the difference in length between two successive pieces which are cut is 10 cm.
   - **i.** Write the lengths of the shortest three pieces cut by the worker.
   - **ii.** Find the length of the 20th piece, when arranged in ascending order of the length, starting from the shortest piece.
   - **iii.** Show that a 5 m long rod will not be sufficient to cut the 50th piece, when arranged in ascending order of the length.

3. On the day that their school celebrated "Annual Savings Day", Yesmi and Indunil start saving money by putting Rs 100 each into their respective savings boxes. After that, they put money into their savings boxes once a week, on the same day of the week that they started saving money. Yesmi put Rs 10 and Indunil put Rs 5 each week into their respective boxes.
   - **i.** How much does Yesmi have in her savings box in the 5th week?
   - **ii.** How much does Indunil have in her savings box in the 10th week?
   - **iii.** At the end of 50 weeks, both of them open their savings boxes and check the amount that each has saved. How much more money has Yesmi saved than Indunil in the 50 weeks?

4. The seats in an outdoor stadium are arranged for a drama in 15 rows according to a pattern with a common difference, such that the first row consists of 9 seats, the second row of 12 seats, the third row of 15 seats, etc.
   - **i.** How many seats are there in total in the first five rows?
   - **ii.** How many seats are there in the 15th row?
   - **iii.** Show that the 10th row has 4 times the number of seats in the first row, according to this pattern.
   - **iv.** Which row consists of 51 seats?

Miscellaneous Exercise

1. The general terms of a few number patterns are given below.

(a) $3n - 5$ (b) $6n + 5$ (c) $6n - 5$

For each number pattern,

- **i.** write the first three terms.
- **ii.** find the 20th term.
- **iii.** find the $(n - 1)$th term in terms of n.

2. Find the general term of each number pattern given below, given that each has a common difference.

i. $-3, 1, 5, 9, \ldots$ **ii.** $0, 4, 8, 12, \ldots$

iii. $1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$ **iv.** $-6, -3, 0, 3, \ldots$

3. Show that the general term of the number pattern 42, 36, 30, 24, ... with a common difference is $6(8-n)$.

4. Uditha is employed in a private company. His first monthly salary is Rs 25 000. From the beginning of the second year onwards, he receives an annual salary increment of Rs 2400 per month.

i. How much is his monthly salary at the beginning of the second year?
ii. Write separately, Uditha's monthly salary during the first three years.
iii. Write an expression for his salary in the *n*th year in terms of *n*.
iv. Find Uditha's monthly salary in the 5th year, by using the expression obtained in (iii) above.

Summary

Summary

- common difference = any term except the first term – the previous term
- The general term of a sequence is denoted by T_n .
- Any term of a sequence can easily be found by using the general term.