21 Cyclic Quadrilaterals

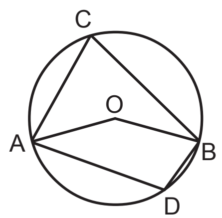
After studying this chapter you will be able to acquire knowledge and the application of :

- proof and application of the theorem and its converse which states that the opposite angles of a cyclic quadrilateral are supplementary.
- proof and application of the theorem and its converse which states that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

21.1 Angles in a circle

We will recall the theorems about angles in a circle which we learned last year.

(1) **Theorem :** The angle subtended at the centre by an arc of a circle is twice the angle subtended by the same arc at the remaining part of the circumference.

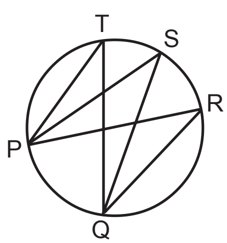


Centre of the circle is O

$$A\hat{O}B = 2\,A\hat{C}B$$

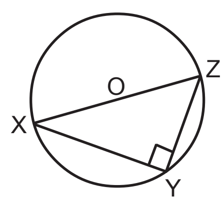
$$A\hat{O}B\ (\text{Reflex}) = 2\,A\hat{D}B$$

(2) **Theorem :**
In a circle, the angles in the same segment are equal



$$P\hat{T}Q = P\hat{S}Q = P\hat{R}Q$$

(3) **Theorem :**
Angle in a semicircle is a right angle



O is the centre of the circle.

$$X\hat{Y}Z = 90^0$$

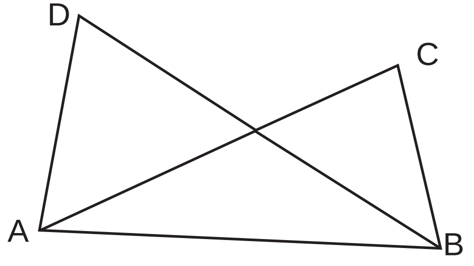
In the above diagrams, the points A,B,C and D and the points P,Q,R,S and T and the points X,Y and Z are points on the same circle.

Thus the points lying on the circumference of a circle are called concyclic.

The converse of the second theorem mentioned above, is a useful theorem for concyclic points.

Theorem :

If a straight line subtends equal angles at two points lying on the same side of the straight line, then the two points and the two end points of the straight line are concyclic.



In the figure given above, the points C and D are on the same side of the straight line AB.

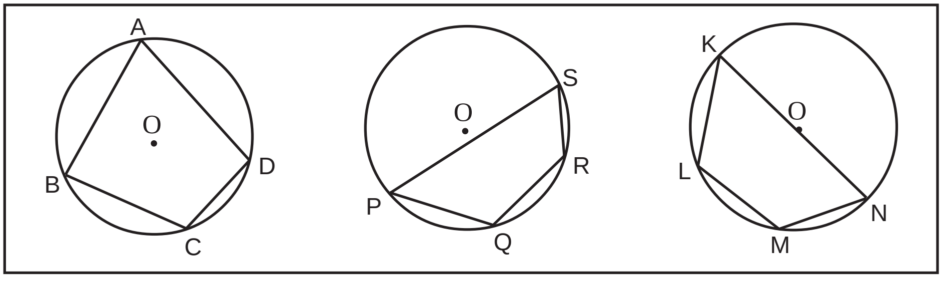
According to the theorem

If $\quad A\hat{C}B = A\hat{D}B$

then A,B,C and D are concyclic.

21.2 Cyclic quadrilaterals

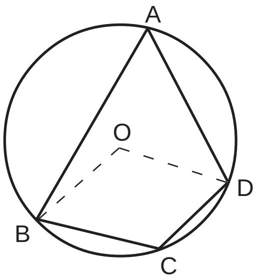
If the four vertices of a quadrilateral lie on the same circle, then the quadrilateral is called a cyclic quadrilateral.



In the above diagram ABCD, PQRS and KLMN are cyclic quadrilaterals.

> **Theorem**
> The opposite angles of a cyclic quadrilateral are supplementary.

Let us prove this theorem



Data :- ABCD is a cyclic quadrilateral. O is the centre of the circle.

T.P.T :- $B\hat{A}D + B\hat{C}D = 180^0$

and $A\hat{B}C + A\hat{D}C = 180^0$

Construction:- Join OB and OD

Proof :- $B\hat{O}D = 2\,B\hat{A}D$ (Angle subtended at the centre is twice the angle subtended at the circumference)

$B\hat{O}D$ (reflex) $= 2\,B\hat{C}D$ (Angles subtended at the centre is twice the angle subtended at the circumference)

$\therefore B\hat{O}D + B\hat{O}D$ (reflex) $= 2\,B\hat{A}D + 2\,B\hat{C}D$

But $B\hat{O}D + B\hat{O}D$ (reflex) $= 360^0$ (Angles at a point)

$\therefore\ 2\,B\hat{A}D + 2\,B\hat{C}D = 360^0$

$B\hat{A}D + B\hat{C}D = 180^0$

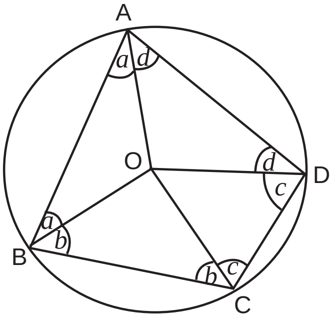
But $A\hat{B}C + A\hat{D}C + B\hat{A}D + B\hat{C}D = 360^0$ (Angles of a quadrilateral)

$\therefore\ A\hat{B}C + A\hat{D}C + 180^0 = 360^0$

$\therefore\ A\hat{B}C + A\hat{D}C = 180^0$

$\therefore$ The opposite angles of a cyclic quadrilateral are supplementary.

This theorem can be proved also by the knowledge of isosceles triangles



Since the radii of a circle are equal the triangles AOB, BOC, COD and AOD are isosceles triangles

$$(a+b)+(b+c)+(c+d)+(d+a)=360^0$$

$$\therefore 2a+2b+2c+2d=360^0$$

$$\therefore (a+b)+(c+d)=180^0$$

$$\therefore A\hat{B}C + A\hat{D}C = 180^0$$

Similarly $(a+d)+(b+c)=180^0$

$$\therefore B\hat{A}D + B\hat{C}D = 180^0$$

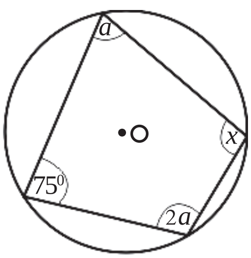
The converse of this theorem too can be used as a theorem. i.e.

If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is a cyclic quadrilateral.

Example 1

Find the angles marked by symbols

(i)



$$x+75^0=180^0$$

$$x=180^0-75^0$$

$$\underline{\underline{x=105^0}}$$

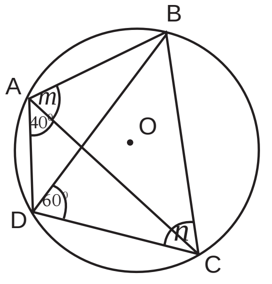
$$a+2a=180^0$$

$$3a=180^0$$

$$\underline{\underline{a=60^0}}$$

$$\underline{\underline{2a=120^0}}$$

(ii)



$m = 60^0$

$B\hat{A}D = 60^0 + 40^0$

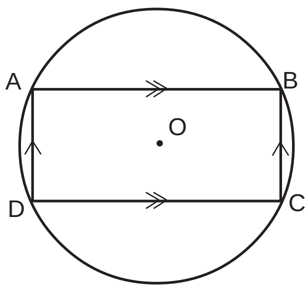
$= 100^0$

$n = 180^0 - 100^0$

$\underline{\underline{n = 80^0}}$

Example 2

The vertices of the parallelogram ABCD are concyclic.
Prove that ABCD is a rectangle.



Data :- The vertices of the parallelogram ABCD are concyclic.

T.P.T :- ABCD is a rectangle

Proof :- Since ABCD is a cyclic quadrilateral

$\hat{B} + \hat{D} = 180^0$ (Opposite angles of a cyclic quadrilateral)

But $\hat{B} = \hat{D}$ (Oppsite angles of parallelogram ABCD)

$\therefore \hat{B} = \hat{D} = 90^0$

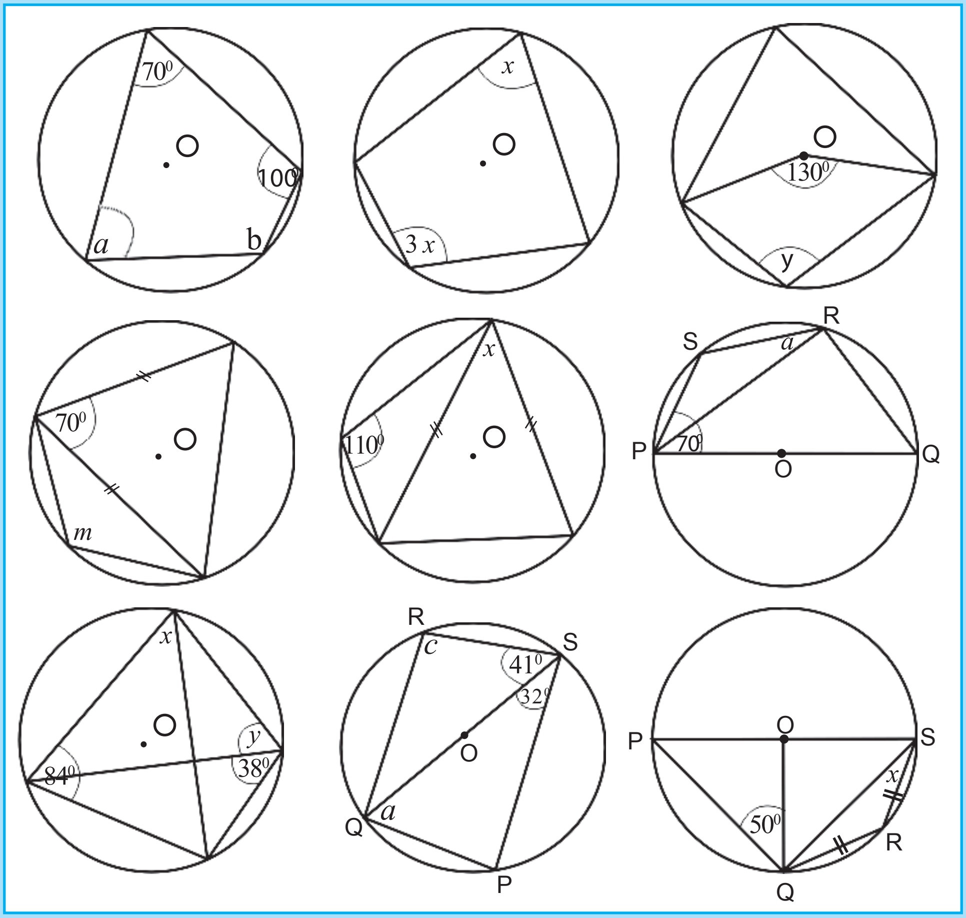
Also ABCD is a parallelogram (Data)

$\therefore$ AB = DC

$\therefore$ ABCD is a rectangle

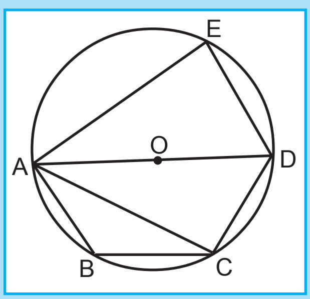
Exercise 21.1

(1) In these digrams O denotes the centre of each circle. Find the angles marked by symbols.



(2) In the diagram, the vertices of the pentagon ABCDE lie on a circle with centre O. AD is a diameter of the circle. If

$D\hat{A}E = 30^0$ and $B\hat{A}C = 25^0$, AB = BC and DC = DE find the magnitudes of the angles of the pentagon.

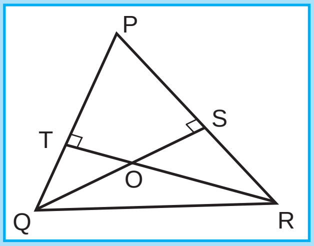


3) In the cyclic quadrilateral ABCD, AB // DC. Prove that $A\hat{B}C = B\hat{A}D$.

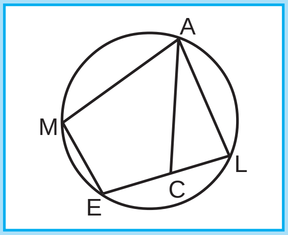
(4) In the cyclic quadrilateral KLMN, $K\hat{L}M = K\hat{N}M$ Prove that KM is a diameter of the circle.

(5) In triangle PQR, the perpendiculars drawn from Q to PR and R to PQ intersect at O. (See figure)
Prove that,

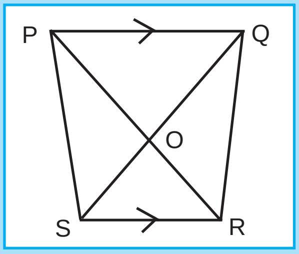
(i) PSOT is a cyclic quadrilateral
(ii) QRST is a cyclic quadrilateral



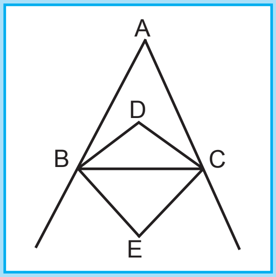
(6) In cyclic quadrilateral MALE, C is a point on EL such that AC = AL. Prove that $A\hat{M}E = A\hat{C}E$.



(7) In quadrilateral PQRS, the diagonals PR and QS intersect at O. If PQ // SR and SO = OR. Prove that P, Q, R and S are concyclic.

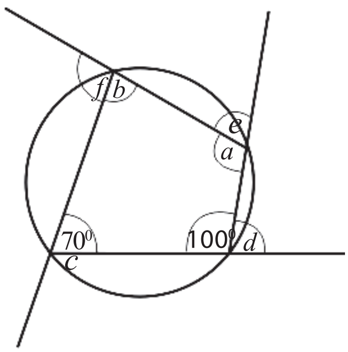


(8) In triangle ABC, the bisectors of the interior angles B and C meet at D and the bisectors of the exterior angles at B and C meet at E. Prove that BDCE is a cyclic quadrilateral.



21.3 The relationship between the exterior angles and the interior angles of a cyclic quadrilateral.

Activity :- 21.1 In this diagram, we will find the angles denoted by symbols.



$a = 180^0 - 70^0 = \ldots$

$b = 180^0 - 100^0 = \ldots$

$c = 180^0 - 70^0 = \ldots$

$d = 180^0 - 100^0 = \ldots$

$e = 180^0 - 110^0 = \ldots$

$f = \ldots - \ldots = \ldots$

We will write the values of interior angles and exterior angles of this cyclic quadrilateral separately.

Interior angles ⟶ $70^0, 100^0, 110^0, 80^0$

Exterior angles ⟶ . . ., . . ., . . ., . . .

According to the values you have got, see whether there is a relationship between the values of the interior angles and exterior angles.

> When considering an exterior angle of a quadrilateral, the angle opposite the interior angle which is adjacent to the exterior angle is called the **interior opposite angle** of the exterior angle mentioned above.

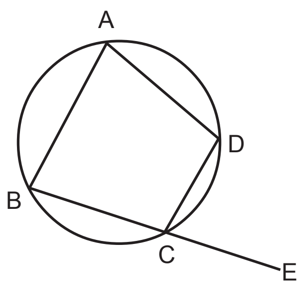
We will complete this table

Value of the exterior angle	110^0	80^0		32^0
Value of the interior opposite angle	110^0		50^0	

The table shows that the value of the exterior angle is equal to the value of the interior opposite angle. This relationship is considered as a theorem in geometry.

Theorem :
In a cyclic quadrilateral, the exterior angle formed by producing a side, is equal to the interior opposite angle.

Proof of this theorem



Data : In cyclic quadrilateral ABCD, side BC is produced to E.

T.P.T. : $D\hat{C}E = B\hat{A}D$

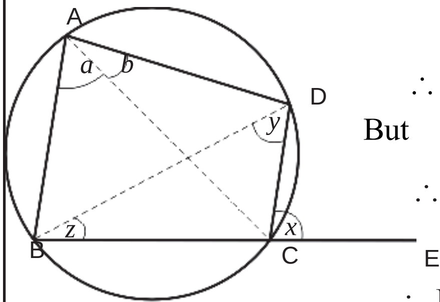
Proof : $B\hat{C}D + D\hat{C}E = 180^0$ (Adjacent angles are supplementry)

$B\hat{C}D + B\hat{A}D = 180^0$ (Opposite angles of a cylie quadrilateral)

$\therefore B\hat{C}D + D\hat{A}C = B\hat{C}D + B\hat{A}D$

$\therefore D\hat{C}E = B\hat{A}D$

This theorem can also be proved by using the theorems on angles in a triangle and angles in a circle.



a = y (Angles in the same segment)
b = z (Angles in the same segment)

$$\therefore\ a + b = y + z$$

But $\quad y + z = x$

$$\therefore a + b = x$$

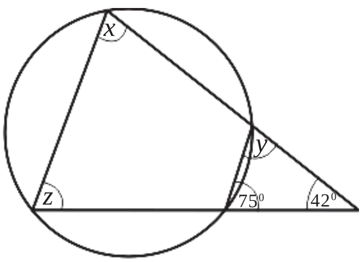
$$x = a + b$$

$$\therefore\ D\hat{C}E = B\hat{A}D$$

The converse of this theorem too can be used as a theorem. Hence if the exterior angle formed by producing a side of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral.

Example 3

Find the values of *x*, *y* and *z* in the figure.



$$\underline{\underline{x = 75^0}}$$

$$y = 180^0 - \left(75^0 + 42^0\right)$$

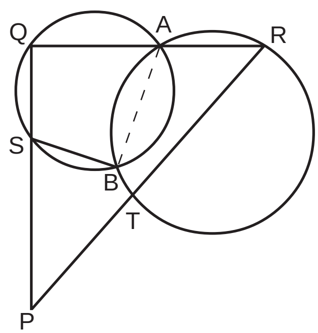
$$y = 180^0 - 117^0$$

$$\underline{\underline{y = 63^0}}$$

$$\underline{\underline{z = 63^0}}$$

Example 4

In the figure, the two circles intersect at A and B. The point A lies on the side QR of triangle PQR. Prove that PSBT is a cyclic quadrilateral.



Data	:-	The two circles are intersect at A and B. The point A lies on the side QR of triangle PQR
T.P.T.	:-	PSBT is a cyclic quadrilateral
Construction	:-	Join AB .
Proof	:-	In the cyclic quadrilateral ABSQ,

$R\hat{A}B = B\hat{S}Q$ (Exterior angle = interior opposite angle)

In the cyclic quadrilateral ABTR,

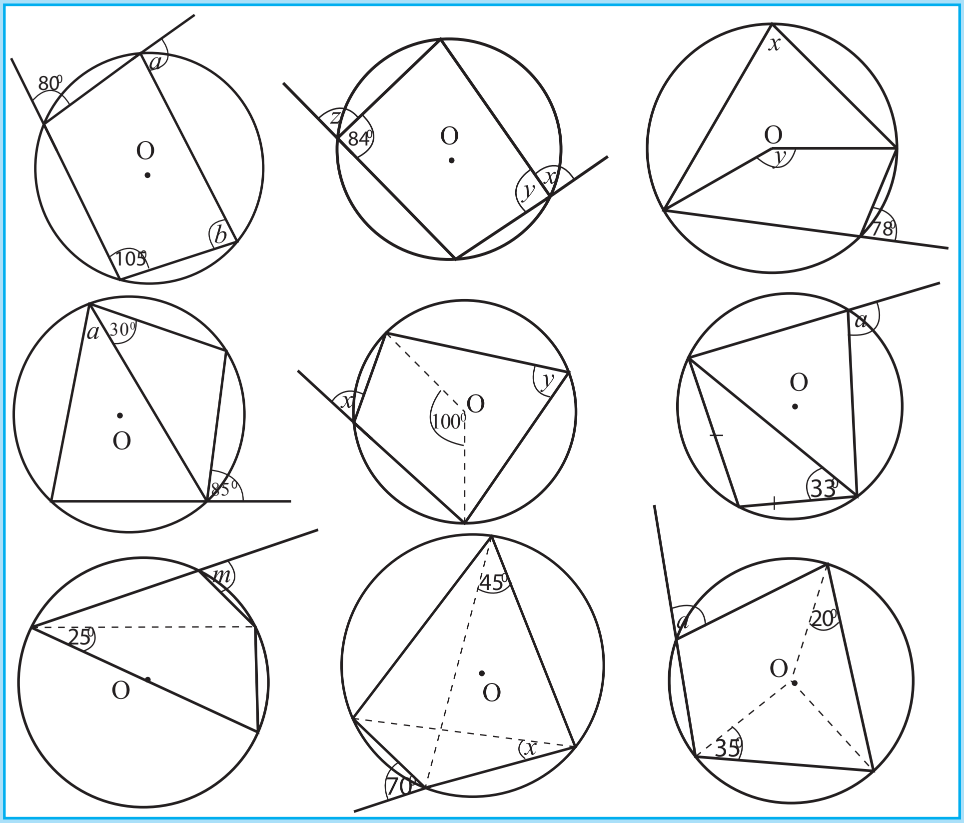
$R\hat{A}B = B\hat{T}P$ (Exterior angle = interior opposite angle)

$\therefore B\hat{S}Q = B\hat{T}P$

PSBT is a cyclic quadrilateral.

Exercise 21.3

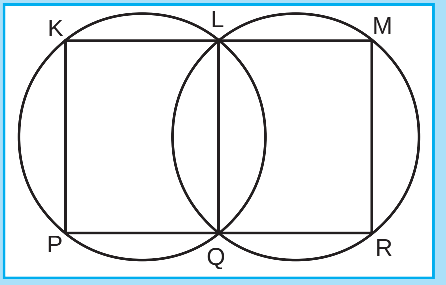
(1) In the following diagrams O denotes the centre of relevant circles. Find the angles denoted by symbols in each circle.



(2) In the given diagram the two circles intersect each other at L and Q. KLM and PQR are two straight lines. Name an angle

(i) equal to angle $K\hat{L}Q$

(ii) supplementary to $K\hat{L}Q$



(3) BC is a diameter of a circle with centre O. Side CD of the cyclic quadrilateral ABCD is produced to E. Using the given diagram

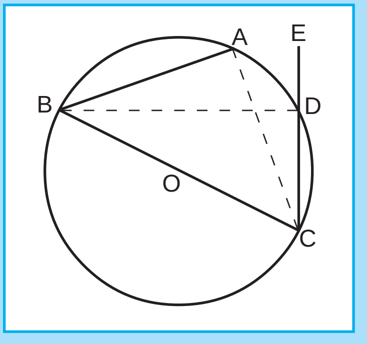
(i) Name a right angle, give reasons.

(ii) Write the relationship between $A\hat{D}E$ and $A\hat{B}C$ give reasons

(iii) If $A\hat{D}E = 75^0$ and $C\hat{A}D = 20^0$

(a) Find the value of $A\hat{B}D$

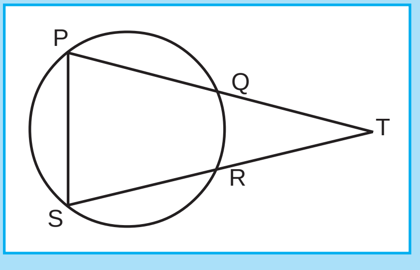
(b) Prove that AB = CD



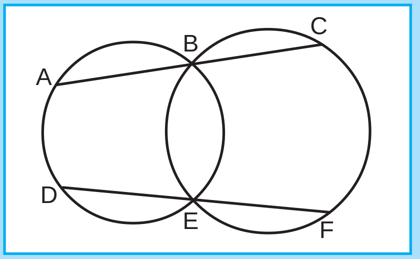
(4) In the cyclic quadrilateral PQRS produced PQ and SR produced meet at T. PT= ST.

(i) Prove that QRT is an isosceles triangle.

(ii) Prove that PS // QR.



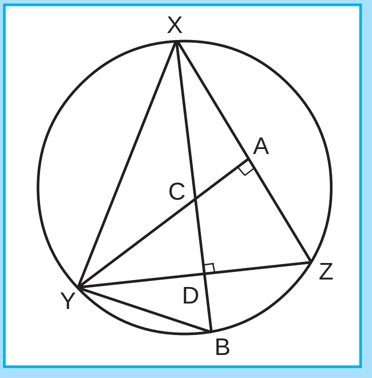
(5) In the figure, the two circles intersect at B and E. ABC and DEF are two straight lines. Prove that AD // CF.



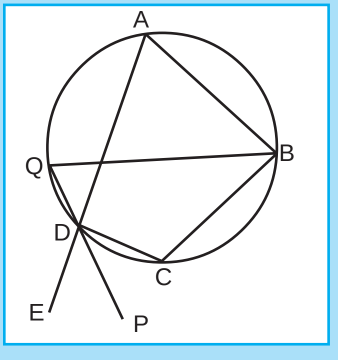
(6) XYZ is a triangle inscribed in a circle. YA is the perpendicular drawn from Y to XZ. The perpendicular drawn from X to YZ meets YA at C and YZ at D and the circle at B.

(i) Prove that ACDZ is a cyclic quadri lateral.

(ii) Prove that $Y\hat{C}B = Y\hat{B}C$



(7) Side AD of the cyclic quadrilateral ABCD is produced to E. The bisector of angle CDE is DP when the bisecter DP produced along the PD direction. it meets the circle at Q. Prove that the line BQ bisects the angle ABC.



(8) KLMN is a cyclic quadrilateral. KL and NM when produced meet at A. KN and LM when produced meet at B. If ABNL is a cyclic quadrilateral, prove that AB is a diameter of the circle passing through the points A, B, N and L.