16 Geometric Progression

After Studing this Chapter you will be able to acquire knowledge and the applications of :

- the concept of the geometric progressions
- the n^{th} term of a geometric progression
- the sum of the first "n" terms of a geometric progression
- the geometric mean

16.1 Introduction

A number sequence is a set of numbers arranged in a definite order. These numbers are called the terms of the sequence and a term next to a term is obtained according to a certain rule. Thus a sequence may consist of a finite number of terms or infinite number of terms.

eg : 3, 6, 12, 24, 48, 96

Let us consider the above number sequence. It contains 6 terms. The first term is 3 and the last term is 96. Every term is obtained by multiplying the previous term by two. So that when every term in the sequence is divided by the previous term the result you get is two.

Consider the following number sequences:

(i) 2, 4, 8, 16, . . .

(ii) 1, 3, 9, 27, 81, . . .

(iii) 64, 32, 16, 8, 4, . . .

(iv) 100, 10, 1, 0.1, 0.01, 0.001, . . .

(v) 81, -27, 9, -3, 1, . . .

Sequence	First term	The constant value obtained by dividing any term by the preceding term
(i)	2	$\frac{4}{2} = \frac{8}{4} = 2$
(ii)	1	$\frac{3}{1} = \frac{27}{9} = 3$
(iii)	64	$\frac{32}{64} = \frac{16}{32} = \frac{1}{2}$
(iv)	100	$\frac{10}{100} = \frac{1}{10} = 0.1$
(v)	81	$\frac{-27}{81} = \frac{9}{-27} = \left(-\frac{1}{3}\right)$

When we divide any term in a number sequence by the preceding term, if the result is a constant value, then such a sequence is called a **geometric progression.** This constant value is called the **common ratio** of the progression.

Notation for the first term of a geometric progression is ***"a"*** and the common ratio is ***"r"***

> A geometric progression of which the first term "*a*" and the common ratio "*r*" can be written as:
>
> $$a, ar, ar^2, ar^3, ar^4, \ldots$$

Example 1

Find the common ratio of each of the following geometric progressions.

(i) 4, 8, 16, 32 . . .
(ii) 64, 32, 16, 8 . . .
(iii) 3, -9, 27, -81. . .
(iv) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54} \ldots$

Solutions:

(i) $r = \frac{8}{4} = \frac{16}{8} = 2$

(ii) $r = \frac{32}{64} = \frac{16}{32} = \frac{1}{2}$

(iii) $r = \frac{-9}{3} = \frac{27}{-9} = -3$

(iv) $r = \frac{1}{6} \div \frac{1}{2} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$

Exercise 16.1

(1) Find the common ratio of each geometric progression given below:

(i) 4, 12, 36, 108. . .

(ii) -3, -9, -27, -81. . .

(iii) 27, -18, 12, -8 . . .

(iv) $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

(v) $\frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{27}{16}, \ldots$

(vi) $x^2, -x, 1, -\frac{1}{x}, \ldots$

(vii) $\frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \frac{2}{27}, \ldots$

(viii) $b^2c^3,\ b^3c^2,\ b^4c,\ b^5, \ldots$

(2) Write the first four terms of each of the following progressions:

(i) $a = 2,\ r = 7$

(ii) $a = 7,\ r = 2$

(iii) $a = 16,\ r = \frac{1}{2}$

(iv) $a = s,\ r = s$

16.2 The n^{th} term of a geometric progression

Let T_n denote the n^{th} term of a geometric progression

Observe the geometric progression 3, 6, 12, 24, . . .
The common ratio is two and the 1^{st} term is 3

$$1^{st} \text{ term} = T_1 = 3 = 3\times 1 = 3\times 2^0$$
$$2^{nd} \text{ term} = T_2 = 6 = 3\times 2 = 3\times 2^1$$
$$3^{rd} \text{ term} = T_3 = 12 = 3\times 4 = 3\times 2^2$$
$$4^{th} \text{ term} = T_4 = 24 = 3\times 8 = 3\times 2^3$$

In a geometric progression of which the first term is "a" and the common ratio is "r",

$$T_1 = a = ar^0 = ar^{1-1}$$
$$T_2 = ar = ar^1 = ar^{2-1}$$
$$T_3 = ar^2 = ar^{3-1}$$

....

....

....

....

....

$$T_{10} = ar^9 = ar^{10-1}$$

The n^{th} term of the progression: $T_n = ar^{n-1}$

> The n^{th} term of a geometric progression of which the first term is "a" and the common ratio is "r" is given by $T_n = ar^{n-1}$

Example 2

Find the 8^{th} term of each of the following geometric progressions:

(i) 3, 6, 12, 24 (ii) 8, 4, 2, 1

Solution

(i) The 1^{st} term, $a = 3$ The common ratio, $r = \frac{6}{3} = 2$

$$n^{th} \text{ term, } T_n = ar^{n-1}$$

$$T_8 = 3 \text{ x } 2^{8-1} = 3 \text{ x } 2^7 = 3 \text{ x } 128 = 384$$

$\therefore$ the 8^{th} term of the progression is 384

(ii) The 1^{st} term, $a = 8$ common ratio, $r = \frac{4}{8} = \frac{1}{2}$

$$n^{th} \text{ term, } T_n = ar^{n-1}$$

$$T_8 = 8 \times \left(\frac{1}{2}\right)^{8-1} = 8 \times \left(\frac{1}{2}\right)^7 = 8 \times \frac{1}{128} = \frac{1}{16}$$

$\therefore$ the 8^{th} term of the progression is $\frac{1}{16}$

Example 3

In a geometric progression the first term is 24, the fourth term is -81 find the common ratio of the progression.

Solution

$$T_1 = a = 24 \quad \text{——— } (1)$$

$$T_4 = ar^3 = -81 \quad \text{——— } (2)$$

$$\frac{(2)}{(1)}, \frac{ar^3}{a} = \frac{-81}{24}$$

ie. $r^3 = \frac{-81}{24} = \frac{-27}{8}$

$$r^3 = \left(\frac{-3}{2}\right)^3$$

$$r = \frac{-3}{2}$$

The common ratio of the progression is $-\frac{3}{2}$

Example 4

The third term of a geometric progression, is 360 and the 6th term is 1215. Find the common ratio and the first term.

$$T_3 = ar^2 = 360 \rightarrow (1)$$

$$T_6 = ar^5 = 1215 \rightarrow (2)$$

$$\frac{(2)}{(1)}, \frac{ar^5}{ar^2} = \frac{1215}{360}$$

$$r^3 = \frac{27}{8}$$

$$r^3 = \left(\frac{3}{2}\right)^3$$

$$r = \frac{3}{2}$$

by substituting $r = \frac{3}{2}$ in equation (1)

$$ar^2 = 360$$

$$a \times \frac{3^2}{2^2} = 360$$

$$a \times \frac{9}{4} = 360$$

$$\therefore a = \frac{360 \times 4}{9}$$

$$\underline{\underline{a = 160}}$$

The 1st term of the Progression is 160 and the common ratio is $\frac{3}{2}$.

Example 5

The first term of a geometric progression is 4 and the sum of the first three terms is 28. Find the first three terms of the progression.
If the first term of the progression is "*a*" and the common ratio is "*r*",

$$a = 4 \rightarrow \quad (1)$$

$$a + ar + ar^2 = 28 \rightarrow \quad (2)$$

From (2) $a(1 + r + r^2) = 28$

by substituting $a = 4$ in (2)

$$4(1+r+r^2) = 28$$

$$1+r+r^2 = \frac{28}{4}$$

$$= 7$$

$$\therefore r^2 + r + 1 - 7 = 0$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$\therefore r = -3 \text{ or } r = 2$$

Such *r* has two values, there exist two progressions,
as $4,\ -12,\ 36, \ldots$ when r = -3 and $4,\ 8,\ 16,\ \ldots$ when r = 2.

Exercise 16.2

(1) Copy the given table and fill in the blank cages, with respect to geometric progressions given below.

(i) 5, 15, 45, 135, 405, 1250, 3645

(ii) 3, 6, 12, 24, 48, 96, 192

(iii) 96, 48, 24, 12, 6, 3, 1.5

(iv) 1000, 100,10, 1, 0.1, 0.01, 0.001

(v) $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \frac{64}{729}$

Progression	a	r	T_1	n	l	T_5
(i)						
(ii)						
(iii)						
(iv)						
(v)						

('n' and 'l' represent the number of terms and the last term respectively)

(2) Find the term stated in each of the geometric progressions given below

(i) 12, 6, 3, . . . T_5

(ii) 1, 2, 4, 8, . . . T_7

(iii) 9, 3, 1, . . . T_6

(iv) 27, 18, 12, . . . T_6

(v) 4, −12, 36, . . . T_7

(3) Find the 1st term and the common ratio of each of the given geometric progression

(i) $T_2 = 4; \quad T_5 = 108$

(ii) $T_3 = 6; \quad T_7 = 96$

(iii) $T_3 = 8; \quad T_9 = 64$

(iv) $T_2 = 6; \quad T_5 = 48$

(v) $T_3 = 40; \quad T_6 = 625$

(4) The sum of the 1st term and the 2nd term of a geometric progression is $\frac{3}{4}$ and the sum of the 3rd term and the 4th term is $\frac{3}{16}$, Find the common ratio. Show that there are two such progressions. State the first four terms of the progression whose common ratio is positive.

(5) The sum of the 1st term and 3rd term of a geometric progression is 20, the second term is 4 more than the 1st term. Write down the first four terms of the progression.

(6) In a geometric progression consisting of 4 terms, the sum of the first two terms is 8 and the sum of the next two terms is 72. Show that there are two progressions which satisfy the above conditions. Write the two progressions.

(7) In a geometric progression, the n^{th} term and the $(n-1)^{th}$ term are 640 and 320 respectively. If the first term is 10, find n.

16.3 Geometric mean

Geometric mean of two positive numbers can be defined as follows. Consider a and b as any two positive numbers. Then the geometric mean between a and b is given as $\sqrt{ab}$. This is a positive quantity.

Consider the geometric progression 3, 6, 12, 24, 48, . . . According to the above relation, the geometric mean between 3 and 12 is $\sqrt{3\times 12}$.

ie. geometric mean between 3 and 12 $= \sqrt{3\times 12}$

$$= \sqrt{36}$$

$$= \underline{\underline{6}}$$

Observing the progression given above, it is clear that 6 lies between 3 and 12. This means that the two given numbers and the geometric mean are consecutive terms in a geometric progression.

If a and b are any two positive numbers and c is the geometric mean between a and b then a, c and b are consecutive terms in a geometric progression.

Consider the geometric progression 125, 25, 5, 1...

The geometric mean between 125 and 5 is $= \sqrt{125\times 5}$

$$= \sqrt{625}$$

$$= \sqrt{25^2}$$

$$= \underline{\underline{25}}$$

Also, 125, 25, and 5 are consecutive terms in a geometric progression

Consider the geometric progression 12, 6, 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, . . .

geometric mean between 6 and $\frac{3}{2}$ $= \sqrt{6 \times \frac{3}{2}}$

$= \sqrt{3 \times 3}$

$= \sqrt{9}$

$= \underline{\underline{3}}$

6, 3 and $\frac{3}{2}$ are consecutive terms in a geometric progression.

Now we will try to find the values of *x* such that when *a* and *b* are any two given numbers and *a*, *x*, *b* are consecutive terms in a geometric progression.
As *a*, *x*, *b* are consecutive terms in a geometric progression, by considering the common ratio

$$\frac{x}{a} = \frac{b}{x}$$

$$x^2 = ab$$

ie. $x = \pm\sqrt{ab}$

$\therefore$ *x* can take the value $+\sqrt{ab}$ or $-\sqrt{ab}$

Accordingly,

when *x* is $+\sqrt{ab}$ we get the progression

$$a, \ \sqrt{ab}, \ b$$

and when *x* is $-\sqrt{ab}$, we get the progression

$$a, \ -\sqrt{ab}, \ b$$

Thus we get two geometric progressions.

Note :

When calculating $\sqrt{ab}$, the number inside the square root sign should definitely be a positive number. To satisfy this either *a* and *b* both should be positive or both should be negative.

Example 6

Find the values of x such that 4, x, 16 are the consecuctive terms in a geometric progression.

Accordingly $\dfrac{x}{4} = \dfrac{16}{x}$

$$\therefore x^2 = 64$$

$$\therefore x = \pm\sqrt{64} = \pm 8$$

$\therefore$ The geometric mean of the number 4 and 16 is $\underline{\underline{\pm 8}}$

Thus we get two geometic progressions such as

4, 8, 16, ... and 4, -8, 16, ...

Example 7

Find a number between 2 and 8 such that the three terms are in a geometric progression. Let us consider that the geometric mean is x then the geometrical progression is 2, x , 8.

$$\frac{x}{2} = \frac{8}{x}$$

$$x^2 = 16$$

$$\therefore x = \pm\sqrt{16} = \pm 4$$

$\therefore$ The progression is $\underline{\underline{2, 4, 8 \text{ or } 2, -4, 8}}$

Example 8

Obtain these numbers between 2 and 32 such that they are five terms of a geometric progression.

2, ..., ..., ..., 32

$T_1 = a = 2; \rightarrow ①$ $T_5 = ar^4 = 32 \rightarrow ②$

$\dfrac{②}{①}$ $\dfrac{ar^4}{a} = \dfrac{32}{2} = 16$

$$r^4 = 16$$

$$\therefore r = \pm 2$$

when r = 2, the three numbers are 4, 8, 16, . . .
Or when r = -2, the three numbers are -4, 8, -16, . . .

Example 9

Find two numbers in between $-\frac{1}{4}$ and 2. Such that they will form a geometric progression.

Here we have to find the relevant numbers to fill the blanks so that $-\frac{1}{4}, \ \dots \ , \dots , 2$ are in a geometric progression.

$$T_1 = a = -\frac{1}{4} \text{———} (1)$$

$$T_4 = ar^3 = 2 \text{———} (2)$$

$$\frac{(2)}{(1)}, \quad \frac{ar^3}{a} = \frac{2}{(-1/4)}$$

$$r^3 = 2 \div \left(-\frac{1}{4}\right)$$

$$= 2 \times (-4)$$

$$= -8$$

$$\therefore r = -2$$

$$\therefore T_2 = ar = -\frac{1}{4} \times (-2) = \frac{1}{2}$$

$$T_3 = ar^2 = -\frac{1}{4} \times (-2)^2 = -\frac{1}{4} \times 4$$

$$= -1$$

$\therefore$ progression is $-\frac{1}{4}, \frac{1}{2}, -1, 2.$

Exercise 16.3

(1) Find the geometric mean between 1 and 64

(2) Find the geometric mean between $\frac{1}{x^2y^2}$ and x^2y^2

(3) A geometric progression can be obtained by placing 4 numbers in between $\frac{1}{8}$ and 128 Find the four numbers.

(4) The geometric mean of two numbers is 12. If one number is 6 find the other number.

(5) If a geometric progression is obtained by including 2 terms between 0.3 and 8.1, find the two terms.

(6) Find three numbers between $3\frac{3}{4}$ and 60 so as to obtain a geometric progression with these two numbers.

When the numbers *a* and *b* are considered, the geometric mean between them is $\sqrt{ab}$. Also when a, *x*, b are consecutive numbers in a geometric progression the values for *x* will be either $-\sqrt{ab}$ or $+\sqrt{ab}$. Each these values will be the geometric mean.

16.4 The sum of the first *n* terms of a Geometric Progression (S_n)

Example 10

A geometric progression of which common ratio is 3, the first term is 2. Find the sum of,

(i) first 3 terms
(ii) first 5 terms
(iii) first 8 terms

Hence form a formula for the sum of the first n terms (S_n)

Solution

(i) Let us write down the terms of this progression in the form of indices.

$$2,\ 2\times 3,\ 2\times 3^2,\ 2\times 3^3,\ \ldots\ldots$$

The sum of the first three terms of the progression is S_3

$$S_3 = 2 + 2(3) + 2\left(3^2\right) —— (1)$$

Multiply the equation (1) by common ratio 3

$$3S_3 = 2\ (3) + 2\ (3^2) + 2\ (3^3) ——— (2)$$

(2) - (1), $3S_3 - S_3 = 2\left(3^3\right) - 2 —— (3)$

All the other terms will vanish leaving only the 1st term and the last term.

From (3)

$$(3-1)S_3 = 2\left[\left(3^3\right)-1\right] = 2\left(3^3-1\right)$$

$$S_3 = \frac{2\left(3^3-1\right)}{(3-1)} —— (A)$$

$$S_3 = 2\left(\frac{26}{2}\right) = \underline{\underline{26}}$$

(ii) Similarly $\qquad S_5 = 2 + 2(3) + 2(3)^2 + 2(3)^3 + 2(3)^4 \quad —— \quad (1)$

(1) $\times 3$, $\qquad 3S_5 = 2(3) + 2(3)^2 + 2(3)^3 + 2(3)^4 + 2(3)^5 \quad —— \quad (2)$

$(2)-(1)$, $\qquad 3S_5 - S_5 = 2(3)^5 - 2$

$$S_5(3-1) = 2\left[(3)^5 - 1\right]$$

$$\therefore S_5 = \frac{2\left(3^5-1\right)}{(3-1)} \quad ——— \quad (B)$$

$$\therefore S_5 = 2\left(\frac{242}{2}\right) = \underline{\underline{242}}$$

(iii) Similarly

$$S_8 = 2+2(3)+2(3)^2+2(3)^3+2(3)^4+2(3)^5+2(3)^6 \ +2(3)^7$$

$$3S_8 = 2(3)+2(3)^2+2(3)^3+2(3)^4+2(3)^5+2(3)^6+2(3)^7 \ +2(3)^8$$

$$S_8 = \frac{2(3^8-1)}{(3-1)} \quad \text{———} \ (C)$$

$$S_8 = \frac{2(6560)}{2} = \underline{\underline{6560}}$$

Accordingly to the pattern in A, B and C above, the sum of the first n terms can be written as,

$$S_n = \frac{2(3^n-1)}{3-1}$$

Let us consider now a geometric progression in the general form.

The sum of the first n terms of a geometric progression with the 1st term a and common ratio r,

$$S_n = a+ar+ar^2+\ldots ar^{n-1} \longrightarrow (1)$$

$$(1)\times r \qquad rS_n = ar^2+ar^2+\ldots ar^{n-1}+ar^n \longrightarrow (2)$$

$$(2)-(1) \qquad rS_n - S_n = -a+ar^n$$

$$S_n(r-1) = a(r^n-1)$$

$$\therefore S_n = \frac{a(r^n-1)}{(r-1)}$$

> If S_n is the sum of the 1st n terms of a geometric progression of wich the 1st term "a" and the common ratio "r", then ,
>
> $$S_n = \frac{a(r^n-1)}{(r-1)} \qquad r \neq 1$$

It is easy for calculations to write

$$S_n = \frac{a(r^n - 1)}{(r-1)} \text{ ,when } r > 1$$

and

$$S_n = \frac{a(1 - r^n)}{(1-r)} \text{ ,when } r < 1$$

Example 11

Find the sum of the first six terms of the geometric progression, $3, \ 6, \ 12, \ . . .$

$$a = 3, \ r = \frac{6}{3} = 2, \ n = 6$$

Since $r > 1$ to find the sum, let us apply . $S_n = \frac{a(r^n - 1)}{(r-1)}$

$$S_6 = \frac{3(2^6 - 1)}{(2-1)} = 3 \times 63$$

$$= \underline{\underline{189}}$$

Example 12

The n^{th} term of a geometric progression is $\left(-\frac{1}{2}\right)^n$. Find the first term of this progression and the sum of the first 8 terms.

$$T_n = \left(-\frac{1}{2}\right)^n$$

When n=1, the first term of the progression is obtained.

$$\therefore T_1 = \left(-\frac{1}{2}\right)^1$$

$$= \underline{\underline{-\frac{1}{2}}}$$

Thus the second term is obtained when $n = 2$,

$$T_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$r = \frac{T_2}{T_1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = \frac{1}{4} \times \frac{-2}{1} = -\frac{1}{2}$$

To find the sum, let us apply

$$S_n = \frac{a(1-r^n)}{(1-r)} \qquad \text{(as r< 1)}$$

$$S_8 = \frac{\left(-\frac{1}{2}\right)\left[1-\left(-\frac{1}{2}\right)^8\right]}{\left[1-\left(-\frac{1}{2}\right)\right]}$$

$$S_8 = -\frac{1}{2}\frac{\left[1-\left(\frac{1}{256}\right)\right]}{\left(\frac{3}{2}\right)}$$

$$S_8 = \frac{-\frac{1}{2}\left[\frac{256-1}{256}\right]}{\frac{3}{2}}$$

$$S_8 = -\frac{1}{2} \times \frac{2}{3}\left[\frac{255}{256}\right]$$

$$S_8 = -\frac{1}{3}\left(\frac{255}{256}\right)$$

$$= \frac{-85}{256}$$

Example 13

Find the number of terms that should be taken for the sum of the geometric progression $5, 20, 80, \ldots$ to be 6825.

$$a = 5, \quad r = 4, \quad S_n = 6825$$

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$6825 = \frac{5\left(4^n - 1\right)}{4 - 1}$$

$$6825 = \frac{5\left(4^n - 1\right)}{3}$$

$$\frac{6825 \times 3}{5} = 4^n - 1$$

$$4^n = 4096 = 4^6$$

$$\therefore n = \underline{\underline{6}}$$

Exercise 16.4

(1) The sum of the progression $2, 6, 18, 54, \ldots$ is 728. Find the number of terms in the progression.

(2) In the geometric progression 4, 8, 16, . . . find,
- (i) the common ratio
- (ii) the 8^{th} term
- (iii) sum of the first eight terms

(3) The sum of the first two terms of a geometric progression is 15.The common ratio of the progression is 2. find
- (i) the first term
- (ii) the seventh term
- (iii) the sum of the first seven terms of the progression

(4) The first term of a geometric progression is 5 and the common ratio is 2. find the sum of the first 8 terms of the progression.

(5) The sum of the terms of the geometric progression $2, \ 4, \ 8, \ . \ . \ .$ is 1022. Find the number of terms in the progression.

(6) The 5th term of a geometric progression is 8 and the 3rd term is 4, the sum of the first 10 terms is a positive number. Find

(i) the first term

(ii) the common ratio

(iii) the sum of the first 10 terms of the progression

(7) Find the value of x if $(x+1), (x+3)$ and $(x+7)$ are three consecutive terms of a geometric progression.

(8) The 2nd, 4th and 8th terms of an arithmetic progression are the 1st three terms of a geometric progression. Find the common ratio of the geometric progression.

(9) $(n-6), n, (n+9)$ are the first three terms of a geometric progression. Find n and the sum of the first 10 terms of the progression.

(10) The value of a machine depreciates every year by 10%. If its original price is Rs. 3000, find its value, to the nearest rupee, after 6 years.

(11) The sum of n terms of a geometric progression is $3^n - 1$, for all values of n, find the first three terms of the progression.

(12). Find the sum of the first 5 terms of a geometric progression whose first term is 81 and the common ratio is $\frac{2}{3}$.

(13). Find the sum of the first 20 terms of a geometric progression whose first term is 36 and the common ratio is $\frac{5}{6}$. Use logarithms.

(14). Show that the sum of the first 10 terms is equal to $16x$ of a geometric progression whose first term is 16 and the common ratio is $\frac{4}{3}$. Use logarithms.

$$x = \frac{4^{10} - 3^{10}}{3^{9}}$$

Can zero be a term of a geometric progression ?