14 Graphs

By studying this lesson you will acquire knowledge on the following :

- Solving simultaneous equations using graphs.
- Drawing graphs of functions of the form $y = ax^2 + bx + c$
- Expressing the behaviour of a quadratic function.
- Determining the maximum value, minimum value and the equation of the axis of symmetry.
- Writing the equation of the axis of symmetry, maximum or minimum value and the coordinates of the turning point using the given quadratic equation, without plotting the graph.
- Finding the roots of the given quadratic equation using the graph.

14.1 Solving simultaneous equations

You have learned about graphs in previous grades, not only in mathematics but in various other subjects as well. There were occasions when facts on graphs like pictograms, bar graphs, pie graphs, linear graphs and curved graphs were gathered.

Recall the various methods used previously to solve simultaneous equations. Let us consider in this lesson the method of solving simultaneous equations using graphs.

Example 1 Solve $x - y = 5$

$5x + 2y = 32$

$x - y = 5$ ——— ①

$5x + 2y = 32$ ——— ②

①$\times 2$, $2x - 2y = 10$ ——— ③

② + ③, $7x = 42$

$x = 6$

Substituting $x = 6$ in ①,

$x - y = 5$

$6 - y = 5$

$y = 1$

$\therefore \underline{\underline{x = 6}}$ and $\underline{\underline{y = 1}}$

The above method of finding the solution is called the algebraic method. This pair of equations can be solved by the graphical method too. For this purpose the linear graphs of the two equations should be drawn on the same coordinate plane.

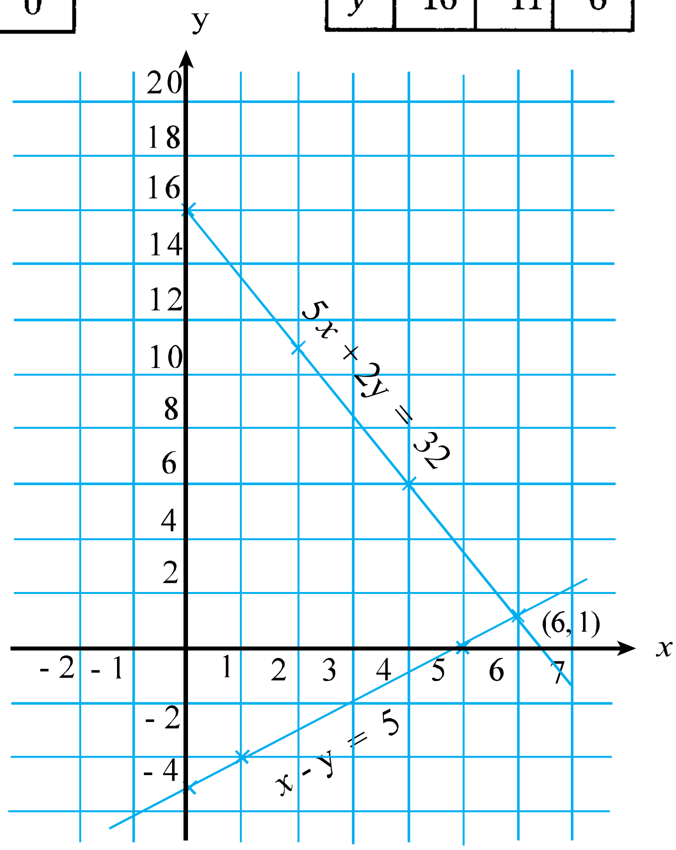
Example 1

$x - y = 5$

x	0	1	5
y	-5	- 4	0

$5x + 2y = 32$

x	0	2	4
y	16	11	6



The coordinates of the point of intersection of the two straight lines is the solution of the given two simultaneous equations. Since the point of intersection is (6,1), the solution is x=6 and y=1. We had the same solution from the algebraic method too.

Let us solve another two simultaneous equations using the graphical method.

Example 2

$$2x - y = 8 \quad \rightarrow (1)$$
$$4x + 3y = 6 \quad \rightarrow (2)$$

(Let us prepare the tables of values to draw the graphs of the two equations.)

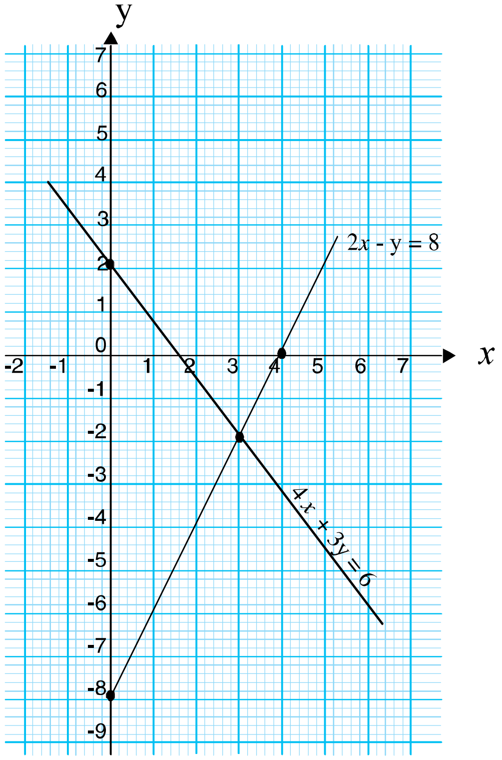
$2x - y = 8$

x	0	4	5
y	-8	0	2

$4x + 3y = 6$

x	0	3	6
y	2	-2	-6

(Let us draw the graphs of the equations.)



The coordinates of the point of intersection of the two graphs are (3,- 2).
The solution of the above pair of simultaneous equations is $x = 3$ and $y = -2$

Exercise 14.1

Solve each pair of simultaneous equations given below and verify the graphical solutions.

(1) $2x + 3y = 9$
$x - y = 2$

(2) $4x + 3y = 6$
$2x - y = 8$

(3) $x - y = 5$
$2x + 3y = 25$

(4) $y - 2x = 3$
$x + y = 6$

(5) $3x - y = -4$
$y - x = 8$

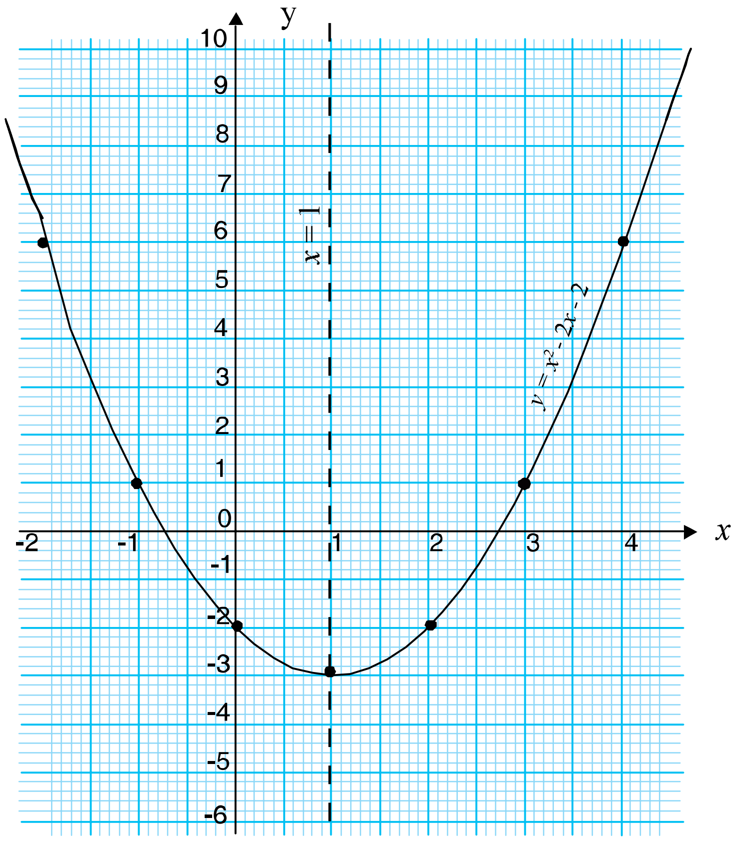
14.2 The graphs of functions of the form $y = ax^2 + bx + c$

You have learnt in grade 10 that the graphs of functions of this form have a minimum value when $a > 0$ and a maximum value when $a < 0$.

Let us draw the graph of the function $y = x^2 - 2x - 2$. For this, prepare a table of values from $x = -2$ to $x = 4$.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	-2	-4	-6	-8
-2	-2	-2	-2	-2	-2	-2	-2
y	6	1	-2	-3	-2	1	6

Prepare a coordinate plane to draw the graph. It is easy to draw the graph taking 10 small squares as 1 unit along the x axis and 10 small squares as 2 units along the y axis. Before drawing the graph, study the dispersion of x and y, and accordingly prepare the coordinate plane, so that the graph paper can be used fully, and then draw the graph accurately.



The following can be observed from the plotted graph

- The graph is a parabola and it is symmetrical about the line $x = 1$
- Therefore the equation of the axis of symmetry of the graph can be written as $x = 1$
- When the value of x increases up to -0.7, the value of y or the value of the function decreases while being positive.
- The value of the function is 0 when the value of x is equal to -0.7
- When the value of x increases from -0.7 to 1, the value of y decreases negatively.
- When the value of x increases from 1 to 2.7 the value of y increases negatively
- When the value of x is 2.7, the value of y is 0
- When the value of x is greater than 2.7, the value of y increases positively.
- The minimum value of the graph is -3.

When the value of x is in between -0.7 and 2.7, y takes negative values. $y<0$ when x takes the values $-0.7< x < 2.7$ so the function is negative when x is between -0.7 and 2.7. When the values of x is less than -0.7 or greater than 2.7, y takes a positive value. That is when $-0.7> x$ and $x > 2.7$ we have $y> 0$. and the function is positive. Accordingly, the roots of the equation $x^2 - 2x - 2 = 0$ can be written as $x = -0.7$ and $x = 2.7$ as the graph intersects the x - axis ($y = 0$) at $x = -0.7$ and $x = 2.7$.

There is a turning point of the graph.It may be a minimum or a maximum.It is also known as the point of refraction. According to the above example the turning point of the graph is a minimum and its coordinates are (1,-3).

Exercise 14.2

Plot the graphs for the equations given below.

(1) a) (i) $y = x^2 - 4$ (Take the values of x from -3 to +3)

(ii) $y = x^2 + 2x - 3$ $(-4 \leqslant x \leqslant +2)$

(iii) $y = x^2 - 4x + 2$ $(-1 \leqslant x \leqslant 5)$

(iv) $y = x^2 + x - 2$ $(-3 \leqslant x \leqslant 3)$

(v) $y = x^2 + x - 1$ $(-4 \leqslant x \leqslant 3)$

(vi) $y = 2x^2 + 3x - 6$ $(-3 \leqslant x \leqslant 3)$

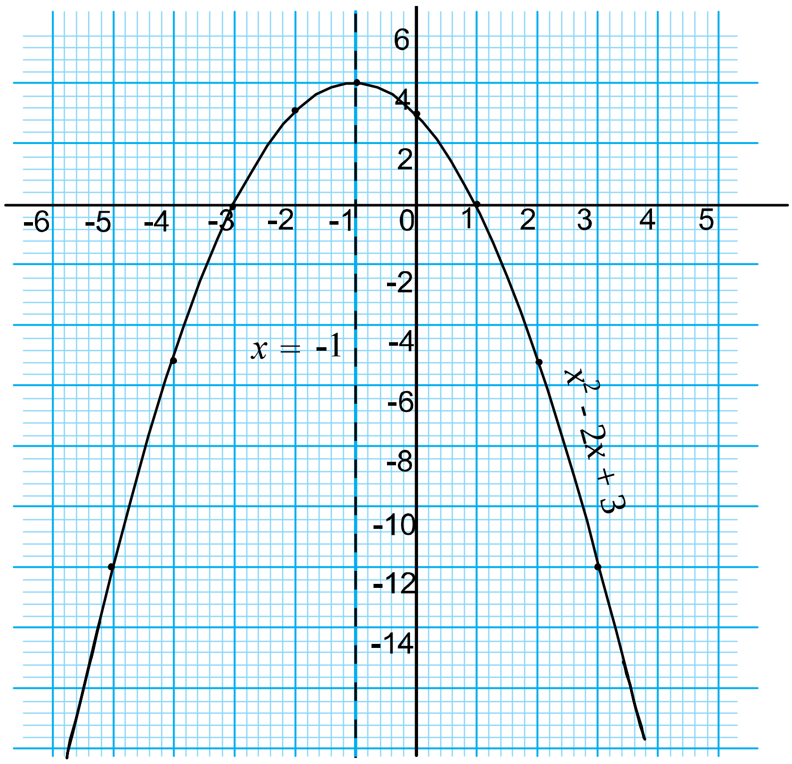
b) • Write the minimum value of each graph
• Write the coordinates of the turning point
• Write the equation of the axis of symmetry

(2) Draw the graph of the function $y = x^2 + 4x - 1$ $(-5 \leqslant x \leqslant 1)$
i. Write the minimum value of the function
ii. Write the equation of the axis of symmetry
iii. Write the coordinates of the turning point
iv. For what range of values of x does the function increase positively?
v. For what range of values of x does the function decrease negatively?
vi. Find the roots of the equation $x^2 + 4x - 1 = 0$

(3) Draw the graph of the function $y = 2x^2 - x - 3$. Using it answer the questions (i), (ii), (iii), (iv) and (v) of question (2) above.

Let us draw a graph of the form of $y = ax^2 + bx + c$ when $a < 0$.
Let us find the values of $y = -x^2 - 2x + 3$ within the range $-5 \leq x \leq 3$

x	-5	-4	-3	-2	-1	0	1	2	3
$-x^2$	-25	-16	-9	-4	-1	0	-1	-4	-9
$-2x$	10	8	6	4	2	0	-2	-4	-6
+3	3	3	3	3	3	3	3	3	3
y	-12	-5	0	3	4	3	0	-5	-12



- The equation of the axis of symmetry is $x = -1$.
- Coordinates of the turning point (-1,4).
- The maximum value of the function is 4.
- When the value of x increases till -3, y increases while being negative.
- The value of the function is 0 when $x = -3$.
- When the value of x increases from -3 to -1 the values of y increases positively.
- When the value of x increases from -1 to +1the value of the function decreases positively.
- When the value of x is greater than 1, the value of the function decreases negatively.
- When the value of x is between -3 and +1, the value of the function is positive. The range in which the function is positive is $-3 < x < 1$.
- When the value of x is less than -3 or greater than +1 the function is negative. The range in which the function is negative is $(x<-3, x>1)$
- The graph intersects the line y = 0 (x-axis) at $x = -3$ and $x = 1$. The roots of $-x^2 - 2x + 3 = 0$ are $x = -3$ and $x = 1$.

Exercise 14.3

(1) Draw the graphs of the following functions

(i) $y = -x^2 + 2x + 8$ (Take the values of x from -3 to +5)

(ii) $y = -2x^2 + 4$ $(-3 \le x \le 5)$

(iii) $y = 3 - 2x^2$ $(-3 \le x \le 3)$

(iv) $y = 5 + 2x - x^2$ $(-2 \le x \le 4)$

(v) $y = 1 + x - 2x^2$ $(-3 \le x \le 4)$

(vi) $y = -(x+4)(x-3)$ $(-6 \le x \le 5)$

a) Write the maximum value of the above graphs

b) Write the coordinates of the turning point

c) Write the equation of the axis of symmetry of each graph

(2) Prepare the table of values to draw the graph of $y = 1 - 3x - 2x^2$

(i) Plot the graph on a coordinate plane

(ii) Draw the axis of symmetry and state the equation.

(iii) State the coordinates of the turning point

(iv) State the range of values of x for which the function is positive.

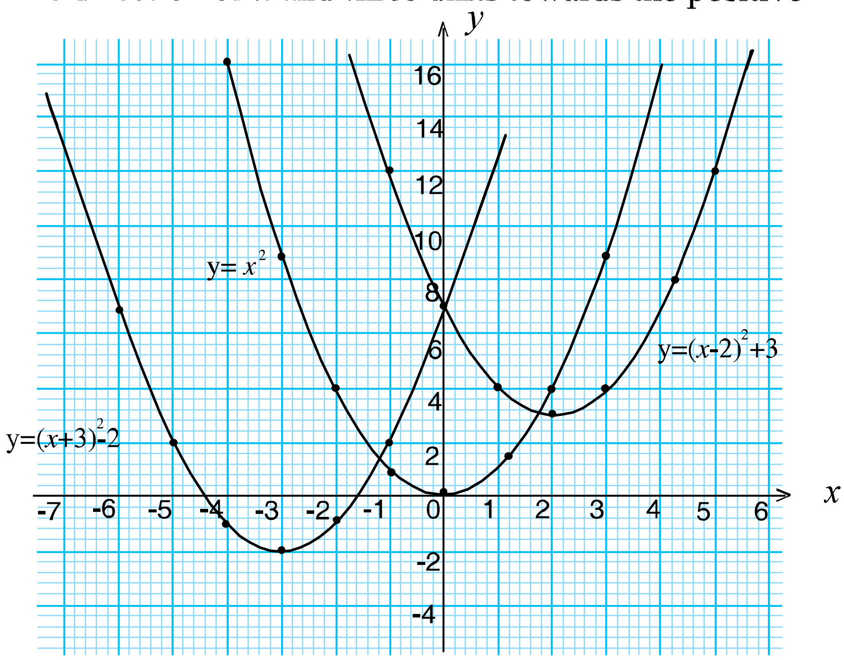
(v) State the roots of $1 - 3x - 2x^2 = 0$.

14.3 Identifying the properties of a graph of a quadratic function without plotting the graph.

Let us consider the graph of a function of the form $y = (x \pm a)^2 \pm b$. Let us draw the graphs of $y = x^2$, $y = (x-2)^2 + 3$ and $y = (x+3)^2 - 2$ on the same coordinate plane.

Copy the graph of the function $y = x^2$, on a tissue paper and place it. on the graphs of $y = (x - 2)^2 + 3$ and $y = (x+3)^2 - 2$. You could see that the graphs coincide with each other.

The function $y = (x - 2)^2 + 3$ is a translation of $y = x^2$, two units towards the positive direction of x and three units towards the positive direction of y.



- The minimum value of $y = x^2$ is 0

  The equation of the axis of symmetry is $x = 0$

  The coordinates of the point of refraction is $(0,0)$

- The minimum value of $y = (x-2)^2 + 3$ is +3

  The equation of the axis of symmetry is $x = 2$

  The coordinates of the point of refraction are $(2,3)$

$y = (x+3)^2 - 2$ The minimum value is -2

The equation of the axis of symmetry is $x = -3$

The coordinates of the point of refraction are $(-3, -2)$

Example 3 : Consider $y = -(x+4)^2 + 3$

The graph of $y = -(x+4)^2 + 3$ has a maximum value.

The equation of the line of symmetry is $x = -4$

The coordinates of the point of refraction are (-4,3)

Let us convert a given quadratic equation of the form $y = x^2 + bx + c$

to the form of $y = (x \pm p)^2 \pm q$

- Consider the function $y = x^2 + 6x + 10$

By completing the squares, it can be written as,

$$y = (x + \tfrac{6}{2})^2 + 1$$

i.e., $y = (x+3)^2 + 1$

Accordingly, comparing this with the general form $y = (x + p)^2 + q$

$p = 3$ and $q = 1$ and the coordinates of the point of refraction are $(-3, 1)$.

Example 4 :

Find the equation of the axis of symmetry, coordinates of the point of refraction, and the maxium value of the function $y = \frac{5}{4} - x - x^2$

In $y = -\left(x + \frac{1}{2}\right)^2 + \frac{3}{2}$ (Coefficient of the squared term is negative)

the maximum value is $\frac{3}{2}$ or 1.5.

Equation of the axis of symmetry is $x = -\frac{1}{2}$ or $x = -0.5$

The coordinates of the turning point are $(-0.5, 1.5)$

Exercise 14.4

(1) Fill in the blanks in the table below without plotting the graphs

Function	Maximum Value	Minimum Value	Equation of the axis of symmetry	Coordinates of the point of inflection
$y=(x+3)^2-2$	...	-2	$x=-3$	$(-3,-2)$
$y=-(x+5)^2+4$	$+4$	...	$x=-5$	$(-5,4)$
$y=\left(x+\frac{3}{2}\right)^2-3$	...	...	...	...
$y=(x+6)-\frac{1}{4}$	...	...	...	...
$y=2\frac{1}{4}-\left(x+\frac{1}{2}\right)^2$	...	...	...	...
$y=x^2-9$	...	...	...	...
$y=x^2+4$	...	...	...	...
...	-3	...	$x=3$	$(3,-3)$
...	...	5	$x=-1$	$(-1,5)$

(2) State the equation of the quadratic function with maximum value $+3$ and axis of symmetry $x=-4$. State also the coordinates of the point of refraction