13 Quadratic Equations

By studying this lesson you will acquire knowledge on the following :

- Solving quadratic equations by factorization
- Solving quadratic equations by completing the square
- Solving quadratic equations using the formula

13.1 Introduction

You have to recall the methods you used in grade 10, in resolving a given expression into factors.

- Identifying the common factors at first if there are any.
- Identifying the factors by grouping the expression first and then finding the common factors.
- Identifying the factors of an expression of the difference of two squares.
- Identifying the factors of a quadratic expression.
- The product of the factors obtained should be equal to the original expression.

Exercise 13.1

Factorise

(1) $x^2 + 7x$ (2) $9x^2 - 18x$ (3) $4y^2 - 25$

(4) $ax - ay - bx + by$ (5) $3x^2 + xy - 6x - 2y$ (6) $x^2 - x - 20$

(7) $x^2 + 15x + 36$ (8) $y^2 - 12y + 20$ (9) $a^2 - 15a - 54$

(10) $2c^2 - c - 6$ (11) $6x^2 + 13x + 6$ (12) $4y^2 - 12y + 9$

(13) $30 + 7y - y^2$ (14) $48 - 19a + a^2$ (15) $24 + k - 3k^2$

(16) $25 + 20p + 4p^2$ (17) $2x^3 - 8x$ (18) $30x^3 + 17x^2 - 2x$

13.2 If the product of several factors of an algebraic expression is zero then at least one factor should be zero.

Applying the concept stated above, the quadratic equations like $3x^2 - 15x = 0$, $y^2 + 3x - 40 = 0$ and $3p^2 - 2p - 5 = 0$ can be solved by means of factors.

Example 1 Solve $3x^2 - 15x = 0$

$3x(x - 5) = 0$

$3x = 0$ or $x - 5 = 0$

$\underline{\underline{x = 0}}$ or $\underline{\underline{x = 5}}$

Example 2 Solve $y(y + 3) = 40$

$y^2 + 3y - 40 = 0$

$(y + 8)(y - 5) = 0$

$y + 8 = 0$ or $y - 5 = 0$

$\underline{\underline{y = -8}}$ or $\underline{\underline{y = 5}}$

Example 3 Solve $\dfrac{3}{2p-1} - \dfrac{2}{3p+2} = 1$

$$\frac{3(3p+2) - 2(2p-1)}{(2p-1)(3p+2)} = 1$$

$(2p - 1)(3p + 2) = 9p + 6 - 4p + 2$

$6p^2 + 4p - 3p - 2 = 5p + 8$

$6p^2 - 4p - 10 = 0$

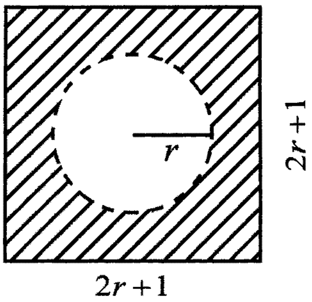
$3p^2 - 2p - 5 = 0$

$(p + 1)(3p - 5) = 0$

$$p+1=0 \quad \text{or} \quad 3p-5=0$$
$$p+1=0 \quad \text{or} \quad 3p=5$$
$$\underline{\underline{p=-1}} \quad \text{or} \quad \underline{\underline{p=1\frac{2}{3}}}$$

Example 4.
A circle with radius r is drawn inside a square of side 2r + 1.

The shaded area of the given diagram is $71\ \text{cm}^2$. Calculate the value of r.



$$(2r+1)^2 - \pi r^2 = 71$$
$$4r^2 + 4r + 1 - \frac{22}{7}r^2 = 71$$
$$28r^2 + 28r + 7 - 22r^2 = 497$$
$$6r^2 + 28r - 490 = 0$$
$$3r^2 + 14r - 245 = 0$$
$$(r-7)(3r+35) = 0$$

$$r-7=0 \quad \text{or} \quad 3r+35=0$$
$$3r=-35$$
$$\underline{\underline{r=7}} \quad \text{or} \quad \underline{\underline{r=-11\frac{2}{3}}}$$

The radius of a circle cannot be a negative value

$\therefore\ r = 7\text{cm}$

Exercise 13.2

Solve the following problems using factors.

(1) $4x^2 = 8x$ (2) $\frac{1}{2}x^2 + 5x = 0$ (3) $y(y+1) = 12$

(4) $a(a-5) = 24$ (5) $p^2 = 4(8-p)$ (6) $24 = c(10-c)$

(7) $15k^2 = 2k + 1$

(8) $6(a-1)(a+1) = 5a$

(9) $4x(x+3) + 9 = 0$

(10) $4x^3 = 25x$

(11) $y(y - 4) = 2(y + 8)$

(12) $\frac{3}{x} - \frac{1}{x-1} = 8$

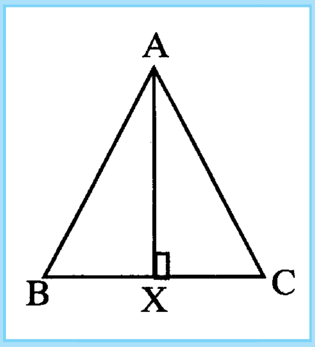
(13) $\frac{4}{a} + \frac{3}{a+1} = 3$

(14) $\frac{2}{p-1} - \frac{3}{p+1} = 1$

(15) The sum of the squares of two positive consecutive odd numbers is 74. Find the two numbers

(16) When four times the reciprocal of a number is added to the number, the result is 4. Find the number.

(17) The length of the base BC of the given ΔABC is $3p$ cm, while the altitude AX is $(p+3)$ cm. If the area of the triangle is 27 cm^2,



(i) form a quadratic equation in terms of p.

(ii) find p and state the length of the base BC.

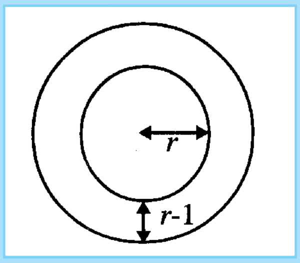
(18) The length of a rectangular table top is $\frac{1}{2}$ m greater than its width. The area of the table top is 5m^2. If the width is "a" metres

(i) find the length in terms of a

(ii) find the value of a

(iii) What is the perimeter of the table top ?

(19) The difference between the areas of the two concentric circles given in the figure is $50\frac{2}{7}$ cm^2. Find the value of r using the given measurements.



(20) The lengths of the sides of a right angled triangle are $x, x-7$ and $x+1$ in centimeters.

(i) Arrange the lengths in ascending order.
(ii) Show that $x > 8$ cm
(iii) By using Pythagoras theorem form a quadratic equation in terms of x
(iv) Solve the above equation and find the lengths of the three sides

(21) In the progression 21, 18, 15, ..., the sum of a certain number of initial terms is 84. Using your knowledge of arithmetic progressions,

(i) write a quadratic equation in n where n is the number of terms.
(ii) By solving the above equation show that there are two values that n can take. Give reasons for this.

13.3 Formulation of quadratic equations when the solution is given

Example 5

Give in the form $ax^2 + bx + c = o$ the equation, of which the roots are 2 and $-\frac{1}{3}$

$$x = 2 \quad \text{or} \quad x = -\frac{1}{3}$$

$$x - 2 = 0 \quad \text{or} \quad 3x = -1$$

$$3x + 1 = 0$$

$$\therefore (x-2)(3x+1) = 0$$

$$3x^2 + x - 6x - 2 = 0$$

$$\underline{\underline{3x^2 - 5x - 2 = 0}}$$

Example 6

$(2x+1)^2 = 25$

(i) Solve the above equation without expanding the expression.
(ii) Build up the quadratic equation in x with the answers you got in (i)

(i) $(2x+1)^2 = 25$

Taking the square roots on both sides,

$$2x+1 = \pm\sqrt{25} = \pm 5$$

$$\therefore 2x+1 = 5 \quad \text{or} \quad 2x+1 = -5$$

$$2x = 5-1 \quad \text{or} \quad 2x = -5-1$$

$$x = \frac{4}{2} \quad \text{or} \quad x = -\frac{6}{2}$$

$$\underline{\underline{x = 2}} \quad \text{or} \quad \underline{\underline{x = -3}}$$

or

$$(2x + 1)^2 = 25$$

$$(2x + 1)^2 - 25 = 0$$

$$(2x + 1)^2 - 5^2 = 0$$

$$(2x + 1 - 5)(2x + 1 + 5) = 0$$

$$(2x - 4)(2x + 6) = 0$$

$$2x - 4 = 0 \quad \text{or} \quad 2x + 6 = 0$$

$$x = \frac{4}{2} \quad \text{or} \quad x = \frac{-6}{2}$$

$$x = 2 \quad \text{or} \quad x = -3$$

(ii)

$$x = 2 \quad \text{or} \quad x = -3$$

$$x - 2 = 0 \quad \text{or} \quad x + 3 = 0$$

$$\therefore (x-2)(x+3) = 0$$

$$x^2 + 3x - 2x - 6 = 0$$

$$\underline{\underline{x^2 + x - 6 = 0}}$$

Exercise 13.3

(1) The pairs of solutions of certian quadratic equations are given below, form quadratic equations in the form

$ax^2 + bx + c = 0$ using each pair of solutions.

(i) -3 and 1 (ii) 4 and 3 (iii) -5 and -1

(iv) $\frac{1}{2}$ and -1 (v) $-\frac{2}{5}$ and $\frac{1}{2}$ (vi) $-\frac{2}{3}$ and $-1\frac{1}{2}$

(2) Solve the equation $(x-3)^2 = 16$. Build up the quadratic equation which has the above solutions.

13.4 Perfect quadratic expressions and completing squares

Copy the table below in your exercise book and fill in the blanks. For this, use the facts you learnt in grade 10 under binomial expressions and the examples given above.

$(x+1)^2$	x^2+2x+1
$(x+2)^2$	...
$(x+3)^2$	...
$(x+5)^2$	...

$(x-3)^2$	x^2-6x+9
$(x-4)^2$	...
$(x-5)^2$	...
$(x-6)^2$	...

All the trinomial expressions you got are called perfect **quadratic expressions.** In this table, observe that the constant term is the square of half the coefficient of the linear term.

Accordingly, determine the constant term that should be added so that x^2+6x

becomes a perfect square.

This is $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Then the expression is

$$x^2+6x+9$$

i.e., $(x+3)^2$

Example 7

Find the constant term which should be added so that x^2-5x becomes a perfect square and express it as a perfect square.

The value to be added $=\left(-\frac{5}{2}\right)^2$

Hence the value to be added is $=\frac{25}{4}$

The expression when $\frac{25}{4}$ is added $= x^2-5x+\frac{25}{4}$

That is $\left(x-\frac{5}{2}\right)^2$

Example 8

If $x^2 - kx + p = (x+q)^2$, find p and q in terms of k.

For $x^2 - kx + p$ to be a perfect square

$$p = \left(\frac{-k}{2}\right)^2$$

then $(x^2 - kx + p) = x^2 - kx + \left(\frac{-k}{2}\right)^2$

$$= \left(x - \frac{k}{2}\right)^2$$

$$\therefore \ q = \frac{-k}{2} \ , \ p = \left(\frac{-k}{2}\right)^2$$

i.e., $p = \dfrac{k^2}{4}$ and $q = \dfrac{-k}{2}$

Exercise 13.4

(1) Find the values that should be added to make the following expressions perfect quadratics. Add the particular value and write the expression as a perfect square.

(i) $x^2 + 14x$ (ii) $x^2 - 5x$ (iii) $x^2 + 2ax$

(2) Examine the equations given below and find the values of p and q.

(i) $x^2 + 5x + p = (x+q)^2$ (ii) $x^2 - 7x + p = (x-q)^2$

(iii) $x^2 + 18x + p = (x-q)^2$ (iv) $x^2 - \frac{2}{3}x + p = (x+q)^2$

13.5 Solving quadratic equations by completing squares

Although the quadratic equations that have factors can be solved by factorization, equations like $x^2 - 4x - 8 = 0$ cannot be factorised. So they should be solved by completing squares.

Example 9

Solve $x^2 = 4(x+2)$. Give the answer correct to the second decimal place.

Removing the brackets and writing the term 4*x* on the left hand side

$$x^2 - 4x = 8$$

Adding the square of half the coefficient of *x* to both sides,

$$x^2 - 4x + 4 = 8 + 4$$

Writing as a perfect square,

$$(x-2)^2 = 12$$

> using logarithms
>
> $\sqrt{12} = 3.464$

Taking square roots,

$$x - 2 = \pm\sqrt{12}$$

$x - 2 = 3.464$ or $x - 2 = -3.464$

$x = 2 + 3.464$ or $x = 2 - 3.464$

$= 5.464$ or $= -1.464$

$\underline{\underline{x = 5.46}}$ or $\underline{\underline{x = -1.46}}$

Example 10

Solve $2x^2 + x - \frac{1}{2} = 0$. Give the answer correct to the first decimal place.

> If the coefficient of x^2 is not 1 , divide the equation by its coefficient to make the coefficient of x^2 equal to 1

Writing $-\frac{1}{2}$ on the right hand side,

$$2x^2 + x = \frac{1}{2}$$

Dividing the equation by the coefficient of the quadratic term (i.e., by 2)

$$x^2 + \frac{1}{2}x = \frac{1}{4}$$

Adding the square of half the coefficient of the linear term to both sides,

$$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 = \frac{1}{4} + \left(\frac{1}{4}\right)^2$$

Writing as a perfect square,

$\sqrt{5} = 2.236$

$$\left(x+\frac{1}{4}\right)^2 = \frac{4+1}{16} = \frac{5}{16}$$

Taking square roots,

$$x+\frac{1}{4} = \pm\frac{\sqrt{5}}{4} = \pm\frac{2.236}{4}$$

$$x+\frac{1}{4} = \frac{+2.236}{4} \quad \text{or} \quad x+\frac{1}{4} = \frac{-2.236}{4}$$

$$x = \frac{-1+2.236}{4} \quad \text{or} \quad x = \frac{-1-2.236}{4}$$

$$x = \frac{1.236}{4} \quad \text{or} \quad x = \frac{-3.236}{4}$$

$$x = 0.309 \quad \text{or} \quad x = -0.809$$

$$\underline{\underline{x = 0.3}} \quad \text{or} \quad \underline{\underline{x = -0.8}}$$

Example 11

Solve $3x^2 - 21x + 8 = 0$. Give the answer correct to the second decimal place.

$$3x^2 - 21x + 8 = 0$$

Writing the constant on the right hand side,

$$3x^2 - 21x = -8$$

Dividing by the coefficient of the quadratic term. (i.e. , by 3)

$$x^2 - 7x = \frac{-8}{3}$$

Adding the square of half the coefficient of the linear term to both sides,

$$x^2 - 7x + \left(\frac{7}{2}\right)^2 = -\frac{8}{3} + \frac{49}{4}$$

$\sqrt{9.583}$
$= 3.095$

Writing as a perfect square,

$$\left(x-\frac{7}{2}\right)^2 = \frac{-32+147}{12} = \frac{115}{12}$$

Taking square roots,

$$x - \frac{7}{2} = \pm\sqrt{\frac{115}{12}} = \pm\sqrt{9.583}$$

$$x - 3.5 = \pm 3.095$$

$x - 3.5 = 3.095$ or $x - 3.5 = -3.095$

$x = 3.5 + 3.095$ or $x = 3.5 - 3.095$

$x = 6.595$ or $x = 0.405$

$\underline{\underline{x = 6.60}}$ or $\underline{\underline{x = 0.41}}$

Exercise 13.5

Solve the quadratic equations given below by completing the squares.

(1) $x^2 - 2x = 4$ (Give the answer to the second decimal place)

(2) $x^2 = -2(4x - 5)$ (Give the answer to the second decimal place)

(3) $x\ (4x - 5) = 3$ (Give the answer to the second decimal place)

(4) $2x^2 + 5x + 1 = 0$ (Give the answer to the first decimal place)

(5) $3x^2 - 3x = \frac{1}{2}$ (Give the answer to the first decimal place)

(6) $3 = x(2x + 3)$ (Give the answer to the first decimal place)

(7) $3(x^2 + 1) = -8x$ (Give the answer to the first decimal place)

(8) $1 = 2x(3 - x)$ (Give the answer to the second decimal place)

(9) $\frac{1}{2}x^2 - \frac{1}{3}x - 1 = 0$ (Give the answer to the second decimal place)

(10) $\frac{2}{x + 2} + \frac{1}{2x + 3} = 1$ (Give the answer to the second decimal place)

13.6 Solving quadratic equations using the formula

Solving the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are constants and $(a \neq 0)$ by completing the square.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a} \quad (\text{As } a \neq 0)$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be applied to solve the quadratic equation $ax^2 + bx + c = 0$

The coefficient of x^2 is a
The coefficient of x is b
The constant term is c

Example 12

Using the formula, solve $2x^2 - 6x + 1 = 0$ (give the answer correct to the second decimal place)

$a = 2, \quad b = -6, \quad c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(1)}}{2(2)}$$

$$x = \frac{6 \pm \sqrt{36 - 8}}{4}$$

$$\sqrt{28} = 5.291$$

$$x = \frac{6 \pm \sqrt{28}}{4} = \frac{6 \pm 5.291}{4}$$

$$x = \frac{11.291}{4} \quad \text{or} \quad x = \frac{0.709}{4}$$

$$x = 2.823 \quad \text{or} \quad x = 0.177$$

$$\underline{\underline{x = 2.82}} \quad \text{or} \quad \underline{\underline{x = 0.18}}$$

Example 13

Using the formula, solve $\frac{1}{3x-2} - \frac{1}{2x+3} = 1$ (give the answer correct to the first decimal place)

$$\frac{1}{3x-2} - \frac{1}{2x+3} = 1$$

$$\frac{(2x+3) - (3x-2)}{(3x-2)(2x+3)} = 1$$

$$2x + 3 - 3x + 2 = (3x-2)(2x+3)$$

$$(3x-2)(2x+3) = 2x + 3 - 3x + 2$$

$$6x^2 + 5x - 6 = -x + 5$$

$$6x^2 + 6x - 11 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(6)(-11)}}{2(6)}$$

$$x = \frac{-6 \pm \sqrt{36 + 264}}{12}$$

$\sqrt{300} = 17.32$

$$x = \frac{-6 \pm \sqrt{300}}{12}$$

$x = \frac{-6 + 17.32}{12}$ or $x = \frac{-6 - 17.32}{12}$

$= \frac{11.32}{12}$ or $x = \frac{-23.32}{12}$

$= 0.943$ or $x = -1.94$

$x = \underline{\underline{0.9}}$ or $x = \underline{\underline{-1.9}}$

Exercise 13.6

Solve the equations given below using the formula

(1) $x^2 - 4x - 1 = 0$ (Give the answer correct to the second decimal place)

(2) $x(x+1) = 1$ (Give the answer correct to the second decimal place)

(3) $2x^2 + 7x + 2 = 0$ (Give the answer correct to the second decimal place)

(4) $3x(x-2) - 2 = 0$ (Give the answer correct to the second decimal place)

(5) $x+4=\dfrac{6}{x}$ (Give the answer correct to the first decimal place)

(6) $2x=\dfrac{9}{x}-4$ (Give the answer correct to the first decimal place)

(7) $\dfrac{3}{x-1}-\dfrac{5}{x+1}=1$ (Give the answer correct to the first decimal place)

(8) $\dfrac{3}{2x+1}+\dfrac{1}{x+1}=2$ (Give the answer correct to the second decimal place)