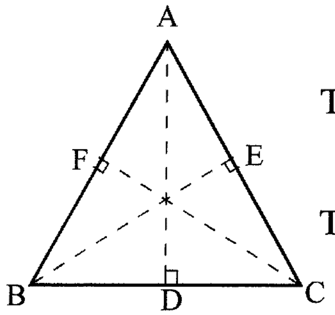
8 The Area of Rectilinear Plane Figures Between Parallel Lines

By studying this lesson you will acquire knowledge on the following :

- Parallelograms on the same base and between the same parallel lines are equal in area.
- Triangles on the same base and between the same parallel lines are equal in area.
- When a triangle and a parallelogram are on the same base and in between the same parallel lines, then the area of the triangle is half the area of the parallelogram.
- Areas of triangles with their bases on the same straight line and having a common vertex are proportional to their bases.

It is useful for you to recall the following facts about the areas of triangles and parallelograms that you have studied in previous grades.



The area of a triangle $= \frac{1}{2} \times \text{base} \times \text{altitude}$

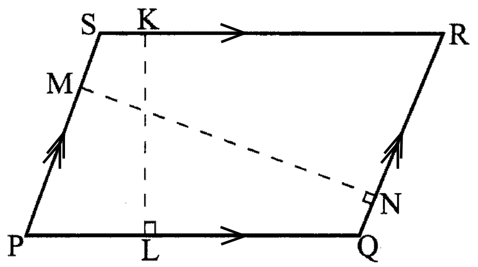
The area of the triangle ABC $= \frac{1}{2} \times BC \times AD$ or

$= \frac{1}{2} \times AC \times BE$ or

$= \frac{1}{2} \times AB \times CF$

The area of a parallelogram = base altitude

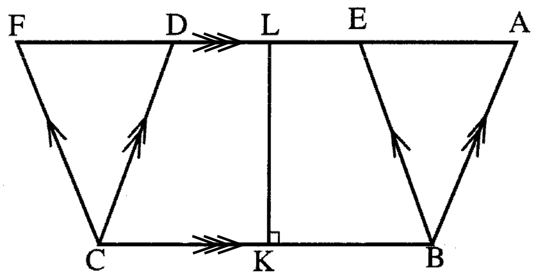
The area of the parallelogram PQRS = PQ × KL or
= SR × KL
= QR × MN or
= PS × MN



8.1 A relationship between the areas of parallelograms

Activity 8.1

Copy down the statements given below and fill in the blanks.

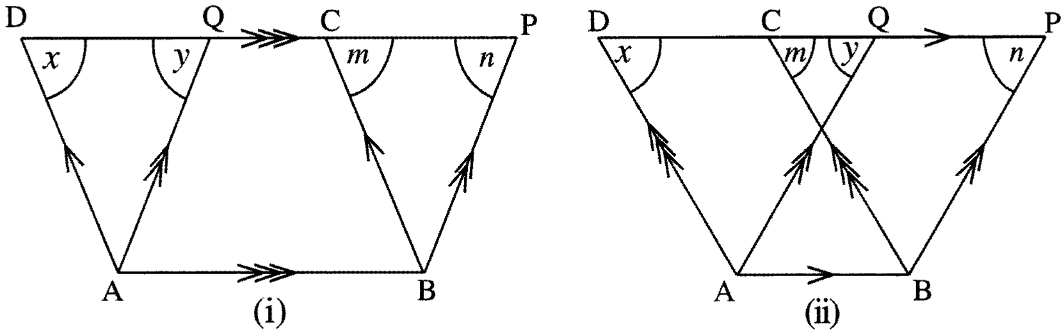


1) There are two parallelograms in the figure named .... and ....
2) There is a common base to these two parallelograms. It is ....
3) KL is the common .... of these two parallelograms.
4) The area of the parallelogram ABCD = CB × ....
5) The area of the parallelogram CBEF = ...× ....
6) According to (4) and (5) above, the areas of the two parallelograms ABCD and CBEF are ....

At the end of this activity you obtained an important relationship. It is put forward as a theorem in geometry as follows.

Theorem: Parallelograms on the same base and between the same pair of parallel lines are equal in area.

Now we will study the formal proof of this theorem.



Data : The parallelograms ABCD and ABPQ are on the same base and between the same pair of parallel lines.

To prove that : The parallelograms ABCD and ABPQ are equal in area.

Proof : To understand this proof easily, some angles in the diagram are named as *x,y,m* and *n*.

In triangles ADQ and BCP

$x = m$ (corresponding angles, AD//BC)

$y = n$ (corresponding angles, AQ//BP)

AD = BC (opposite sides of the parallelogram ABCD)

$\therefore \Delta$ ADQ $\equiv \Delta$ BCP (A, A, S)

$\therefore \Delta$ ADQ and Δ BCP are equal in area too

In figures (i) and (ii) when the area of each triangle is subtracted from the area of the quadrilateral ABPD

ABPD $- \Delta$ ADQ = ABPD $- \Delta$ BCP

$\therefore$ ABPQ = ABCD

In figures (i), When the area of each triangle is added to the area of the quadrilateral ABCQ

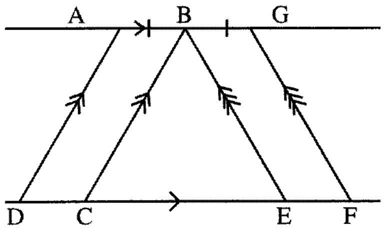
ABCQ + Δ ADQ = ABCQ + Δ BCP

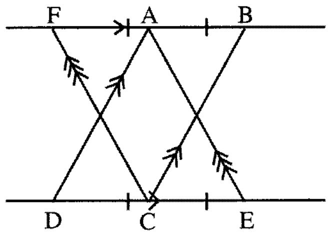
ABCD = ABPQ

$\therefore$ The Parallelograms ABCD and ABPQ are equal in area.

Investigation

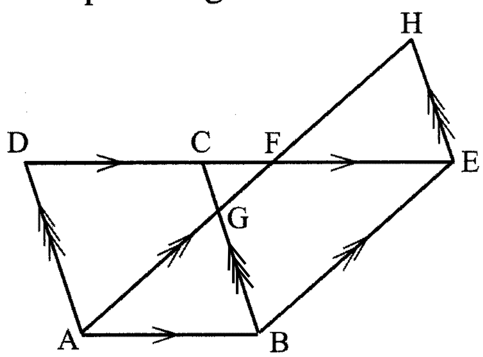
Are the parallelograms between the same parallel lines equal in area only when they are on the same base ?





Example 1

is a straight line. ABCD and ABEF are parallelograms. EH is drawn parallel to BC to meet AF produced at H. The straight lines BC and AH intersect at G. Prove that the parallelograms ABCD and BGHE are equal in area.



Data : The parallelograms ABCD & ABEF are on the same base AB and are in between the same pair of parallel lines AB and DE

To prove that : ABCD and BGHE are equal in area.

Proof : ABCD and ABEF are parallelograms. (data)

$\therefore$ ABCD = ABEF (same base AB and AB//DE)

In the quadrilateral BGHE,

BE//GH (because BE//AF)

BG//EH (because BC//EH is given)

$\therefore$ BGHE is a parallelogram.

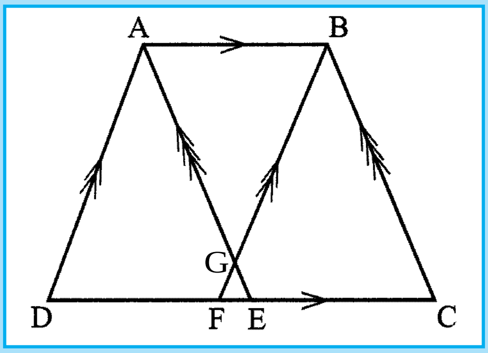
BGHE = ABEF (same base BE and BE // AH)

ABCD = ABEF (proved)

$\therefore$ ABCD and BGHE are equal in area

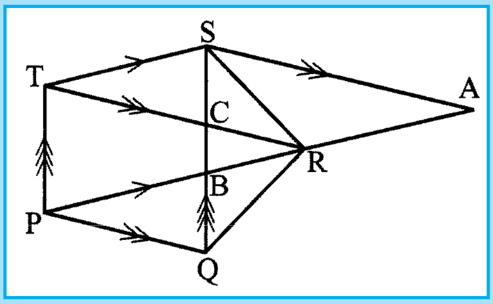
Exercise 8.1

(1)



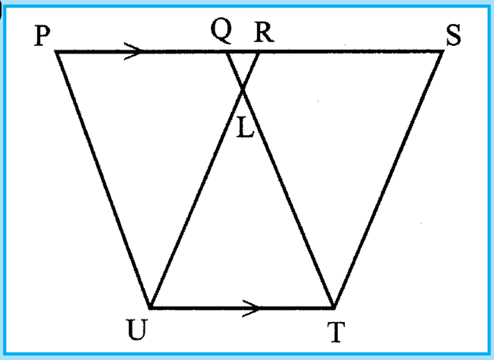
Name two parallelograms in the diagram. Are they equal in area ?

(2)



According to the data given in the diagram name two parallelograms equal in area to the parallelogram PBST.

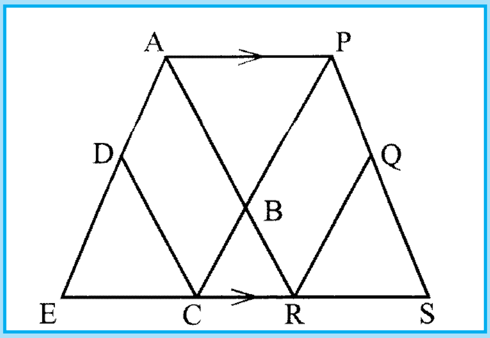
(3)



In the diagram, PS//UT, PU//QT and RU//ST .

I. Copy down the figure and mark the given data.

II. Prove that the quadrilaterals PQLU and RSTL are equal in area.

5) 

In the diagram, ABCD and PQRB are two parallelograms. $ES // AP$. PBC, ADE ABR, PQS are straight lines.

i. Copy down the diagram and mark the given data

Prove that:

ii. APCE and APSR are equal in area.

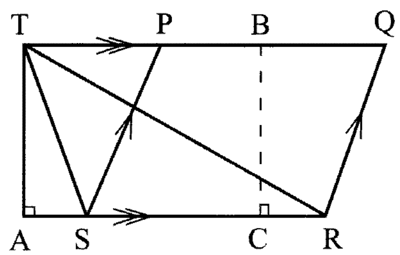
iii. $\Delta CDE \equiv \Delta QRS$

iv. The parallelograms ABCD and PQRB are equal in area.

8.2 The relationship between the area of a parallelogram and a triangle between the same parallel lines and on the same base

Activity 8.2

Fill in the blanks



i. In the diagram SRT is a triangle. PQRS is a.................

ii. There is a common base for this triangle and the parallelogram. That is ....

iii. TA is the altitude of the triangle SRT . The altitude of the parallelogram PQRS is ....

iv. The area of the triangle $SRT = \frac{1}{2} \times SR \times$ ....

v. The area of the parallelogram $PQRS = \text{..........} \times BC$

vi. Because TQ//AR, TA = ..., Therefore the area of the parallelogram

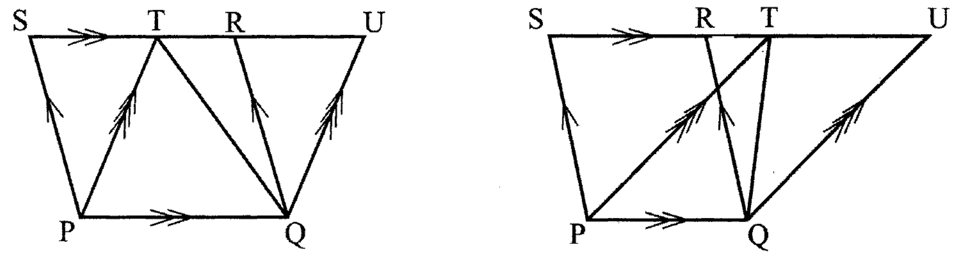
PQRS = SR × TA, and the area of the t rangle SRT= $\frac{1}{2} \times$ SR ×.........

vii. According to (vi) above, the area of the triangle SRT is...... of the area of the parallareloram PQRS.

The relationship obtained at the end of this activity is a theorem in geometry.

> **Theorem:** If a triangle and a parallelogram are on the same base and between the same pair of parallel lines, then the area of the triangle is equal to half the area of the parallelogram.

To understand further about the relationship denoted by this theorem, study the formal proof given below:



Data : The parallelogram PQRS and the triangle PQT are on the same base PQ and between the parallel lines PQ and SR

To prove that : Area of Δ PQT = $\frac{1}{2}$ Area of parallelogram PQRS.

Construction : Complete the parallelogram PQUT.

Proof : Δ PQT = $\frac{1}{2}$ PQUT (because TQ is a diagonal of the parallelogram PQUT)

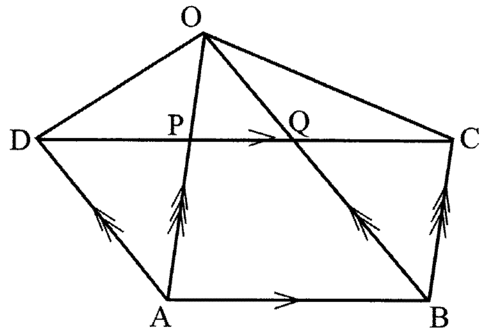
But, PQRS = PQUT (same base PQ and PQ //SU)

$\therefore \Delta$ PQT = $\frac{1}{2}$ PQRS (in area)

> **Investigation**
> If a triangle and a parallelogram are in between a pair of parallel lines should they be on the same base for the area of the triangle to be half the area of the parallelogram?

Example 2

In the trapezium ABCD, AB//DC and DC > AB. AO, the line drawn parallel to BC, and BO, the line drawn parallel to AD, meet at O.Prove that the triangles ADO and BCO are equal in area.



Data : In the trapezium ABCD, AB//DC and DC> AB. AO,the line drawn parallel to BC and BO, the line drawn parallel to AD meet at O.D,,C intersects AO and BO at P and Q respectively.

To prove that: Area of the triangle ADO = Area of the triangle BCO

Proof : In the quadrilateral ABQD

AB//DQ (since AB//DC)

AD//BQ (since AD//BO)

$\therefore$ ABQD is a parallelogram

Similary ABCP is also a parallelogram

$\therefore$ ABQD = ABCP (same base AB, AB//DC)

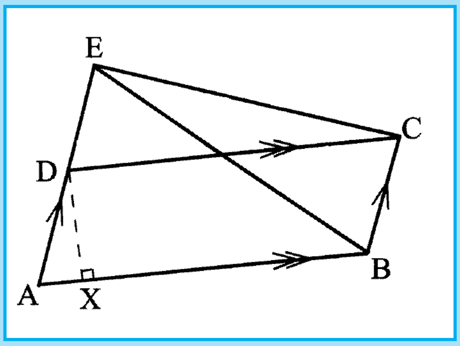
Then $\frac{1}{2}\text{ABQD} = \frac{1}{2}\text{ABCP}$

But, Δ ADO $= \frac{1}{2}$ABQD (same base AD, AD //BO)

Δ BCO $= \frac{1}{2}$ABCP(same base BC, BC//AO)

$\therefore$ Δ ADO = Δ BCO (in area)

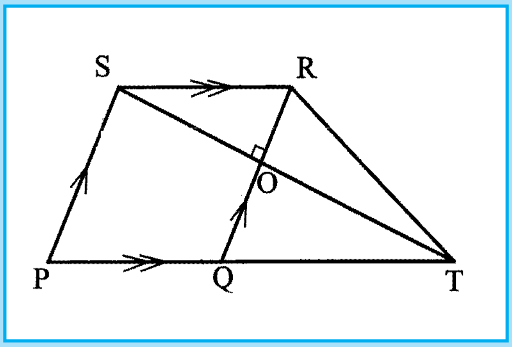
Exercise 8.2



(1) In the parallelogram ABCD, the side AD is produced to E. AB = 7.5 cm and DX = 6 cm. DX $\perp$ AB.

(i) Find the area of the parallelogram ABCD.

(ii) Find the area of the triangle BCE.



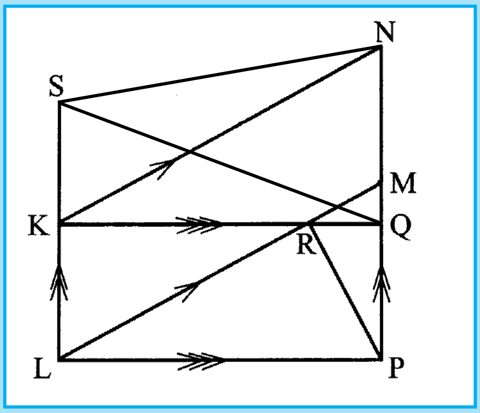
(2) In the parallelogram PQRS, the perpendicular drawn from S to QR and PQ produced meet at T. The lines ST and QR intersect at O.

(i) State the relationship between the area of the triangle RST and the area of the parallelogram PQRS and give reasons for it.

(ii) If PS = 6 cm and SO = 8 cm,find the area of the triangle RST

(3) Construct the triangle ABC in which AB = 5 cm, $A\hat{B}C = 105^0$ and BC = 4.5 cm. Construct the parallelogram ABCE. Is the area of ABCE twice the area of triangle ABC ?

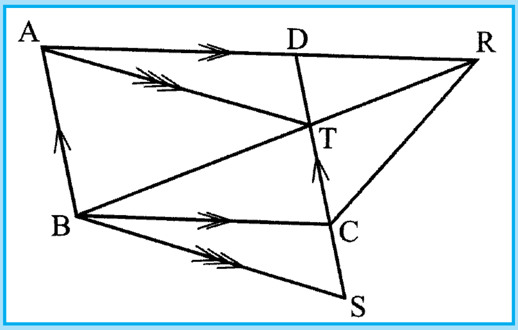
(4) In the diagram,KLMN and KLPQ are two parallelograms. PQMN is a straight line. The lines LM and KQ intersect at R.The side LK is produced to S.



(i) Copy down the diagram and mark the data.

(ii) Prove that the area of the triangle SMN= the area of the triangle LPR.

5) In the diagram,T is on the side DC of the parallelogram ABCD. AD and BT produced meet at R. BS is drawn parallel to AT to meet DC produced at S.



(i) Copy down the diagram and mark the given data

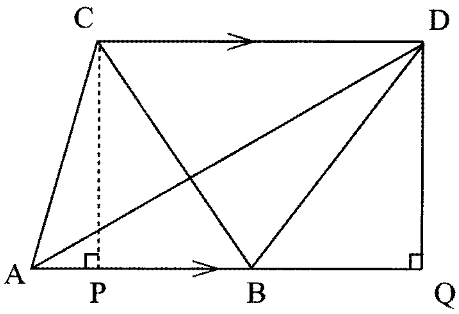
(ii) State the relationship between the areas of the triangle BCR and the parallelogram ABCD and give reasons for it.

(iii) Prove that the area of Δ BSC = area of Δ CTR

8.3 A relationship between the areas of two triangles between the same parallel lines and on the same base.

Activity 8.3

Fill in the blanks



(1) In the diagram, there is a common base to the triangles ABC and ABD. It is..........

(2) The altitude of the triangle ABC is .... and the altitude of the triangle ABD is DQ.

(3) Since AQ//CD, the altitudes CP and DQ are......

(4) The area of the triangle $ABC=\frac{1}{2}\times AB\times$......

(5) The area of the triangle $ABD=\frac{1}{2}\times$........$\times DQ$

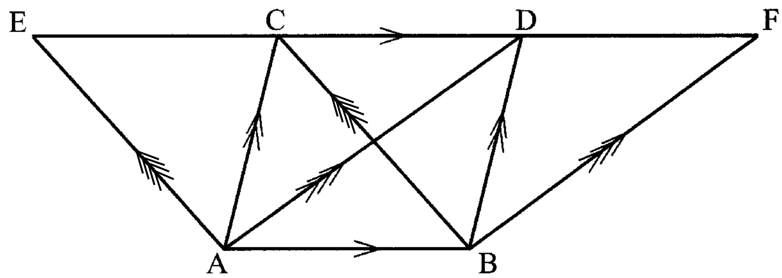
(6) Since CP = DQ, $\frac{1}{2}\times AB\times$....... $=\frac{1}{2}\times$......$\times DQ$

(7) Accordingly, the area of the triangle ABC is ……… to the area of the triangle………

The relationship you obtained at the end of this activity is also an important theorem in geometry. It is stated below.

Theorem: Triangles on the same base and between the same pair of parellel lines are equal in area.

To understand this theorem further, study the formal proof given below.



Date : Triangles ABC and ABD are on the same base AB and between the pair of parallel lines AB and CD.

To prove that : area of Δ ABC = area of Δ ABD

Construction : Complete the parallelogram ABCE and ABFD.

Proof :

$$ABCE = ABFD \text{ (same base AB and AB // EF)}$$

$$\therefore \frac{1}{2}ABCE = \frac{1}{2}ABFD$$

But,

$$ABC\Delta = \frac{1}{2}ABCE \text{ (same base AB and AB//EF)}$$

$$\text{Similarly, } ABD\Delta = \frac{1}{2}ABFD$$

$$\therefore \underline{\underline{ABC\Delta = ABD\Delta}} \text{ (in area)}$$

Investigation

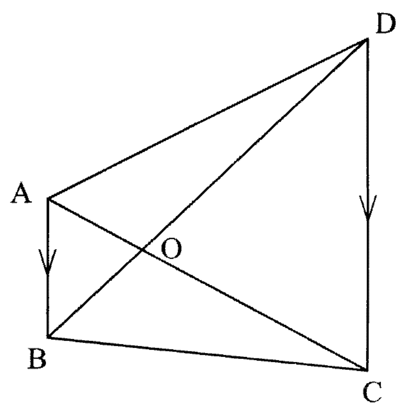
- Are two parallelograms between the same pair of parallel lines, equal in area only if they are on the same base?
- When a triangle and a parallelogram are on the same base and between the same pair of parallel lines, the area of the triangle is half that of the parallelogram. Should the particular triangle and the parallelogram be on the same base?
- Are two triangles between the same pair of parallel lines equal in area only if they are on the same base?

After considering the above questions see whether you can arrive at the conclusions given below.

1. Parallelograms between the same pair of parallel lines and on equal bases are equal in area.
2. If a triangle and a parallelogram are between the same pair of parallel lines and on equal bases, then the area of the triangle is equal to half that of the parallelogram.
3. Triangles between the same pair of parallel lines and on equal bases are equal in area.

Example 3:

In the diagram, AB//DC



(1) Name a triangle equal in area to the triangle ABC and give reasons.

(2) Name a triangle equal in area to the triangle ACD and give reasons.

(3) Prove that the area of ΔBOC = the area of ΔAOD

Answers:

(i) ABCΔ = ABDΔ (same base AB and AB//DC)

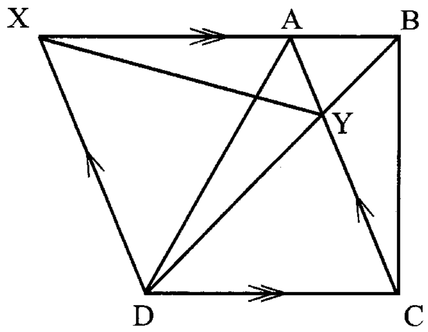
(ii) ACDΔ = BCDΔ (same base DC and AB//DC)

(iii) ABCΔ = ABDΔ (proved above)

$\therefore$ ABCΔ – AOBΔ = ABDΔ – AOBΔ

$\therefore$ BOCΔ = AODΔ (in area)

Example 4:



In the diagram ABCD is a trapezium and AB//DC. DX drawn parallel to CA meets the extended side BA at X. Diagonals AC and BD intersect at Y.

(i) Prove that the area of BYXΔ = the area of ABDΔ

(ii) Prove that the area of BYXΔ = the area of ABCΔ

Answers:

Data : In the diagram,ABCD is a trapezium and AB//DC. DX drawn parallel to CA meets the extended side BA at X. Diagonals AC and BD intersect at Y.

To prove that : (i) The area of BYXΔ = The area pf ABDΔ

(ii) The area of BYXΔ = The area of ABCΔ

Proof : (i) AYXΔ = AYDΔ (same base AY, AY//XD)

$\therefore$ AYXΔ + ABYΔ = AYDΔ + ABYΔ

$\therefore$ BYXΔ = ABDΔ

(ii) BYXΔ = ABDΔ (proved)

ABCΔ = ABDΔ (same base AB and AB//DC)

$\therefore$ BYXΔ = ABCΔ

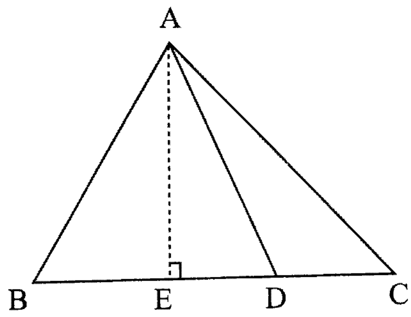
In two triangles which are equal in area under the above theorem,

(i) The same base or equal base should be on one of the two parallel lines.
(ii) The remaining vertex of each triangle should be on the line parallel to the base.

8.4 Relationship between the areas of two triangles of the same height.

Activity 8.4

Fill in the blanks.



(1) Δ ABD and Δ ADC in the diagram have the same altitude. It is..........

(2) The base of each triangle is on the straight line.........

(3) The area of the triangle ABD= $\frac{1}{2} \times BD \times \ldots\ldots$

(4) The area of the triangle ADC= $\frac{1}{2} \times \ldots\ldots\ldots \times AE$

(5) When the area of the triangle ABD is divided by the area of the triangle ADC,

$$\frac{\text{Area of the triangle ABD}}{\text{Area of the triangle ADC}} = \frac{\frac{1}{2} \times BD \times \ldots\ldots\ldots\ldots}{\frac{1}{2} \times \ldots\ldots\ldots\ldots \times AE}$$

$$\therefore \quad \frac{ABD\Delta}{ADC\Delta} = \frac{BD}{\ldots\ldots}$$

So that $\quad ABD\Delta : ACD\Delta = BD : \ldots\ldots\ldots\ldots$

This means that the areas of the triangles ABD and ADC are proportional to the bases of the two triangles. This relationship is also given as a geometrical theorem as follows.

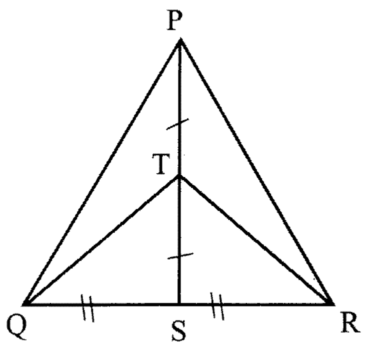
Theorem:	The areas of triangles with a common vertex and with their bases lying on the same straight line are proportional to the length of their bases.

If two triangles with the above properties have equal bases, then the triangles are equal in area.

Hence triangles on equal bases and with the same altitude are equal in area.

Example 5

S is the midpoint of the side QR of triangle PQR. T is the mid point of PS. If the area of the triangle PQR is 60 cm^2, find the area of the triangle PQT.



As
$$QS = SR$$
$$PSQ\Delta = PSR\Delta$$
$$\therefore PSQ\Delta = \frac{60}{2}\text{ cm}^2$$
$$= 30\text{cm}^2$$

As
$$PT = TS$$
$$PQT\Delta = TQS\Delta$$
$$\therefore PQT\Delta = \frac{30}{2}\text{ cm}^2$$
$$= \underline{\underline{15\text{cm}^2}}$$