2 Indices and Logarithms I

By studying this chapter you will acquire knowledge on the following:

- Integral indices
- Rational indices
- Laws of Logarithms (for powers and roots)
- Expressions including roots and powers
- Solving equations including powers and roots (without using log tables.)

Let us recall our knowledge on the multiplication and division of indices, by doing the exercise given below.

Exercise 2.1

Simplify

(1) $x^3 \times x^2 \times x$ (2) $x^4 \times x^2$ (3) $\left(x^4\right)^2$ (4) $a^0 \times a \times a^2$

(5) $(ab)^2 \times a^2 \times b^2$ (6) $\dfrac{\left(3a^2\right)^3 \times \left(a^2\right)^3}{a^5}$ (7) $\left(p^3q^3\right)^2 \times p^2 \times q^3$

(8) $\left(3a^2\right)^3 \div a^5$ (9) $\dfrac{p^6}{\left(p^2\right)^2}$ (10) $\dfrac{2x^2 \times 3x^3}{3x^2 \times 2x^3}$

(11) $\dfrac{x^3 \times x^2}{x^8}$ (12) $\dfrac{2a^0 \times 3a^5}{(2a^3)^2}$

Nimal says that the values of $a^0, \left(a^0\right)^1, \left(a^1\right)^0$ are the same. Do you agree or disagree with him? Give reasons for your answer.

2.1 Fractional Indices

$$a^8 = a^4 \times a^4 = a^5 \times a^3 = a^6 \times a^2$$
$$a^6 = a^3 \times a^3 = a^4 \times a^2 = a^5 \times a^1$$
$$a^4 = a^2 \times a^2 = a^3 \times a^1$$
$$a^2 = a^1 \times a^1$$

Observe that the above powers are divided into two factors. Each power on the left hand side is written as a product of two powers.

The laws of indices learnt before are true for fractional indices too.

When m & n are rational numbers,	When m & n are rational numbers,
$a^n \times a^m = a^{m+n}$	$(a^m)^n = a^{mn}$

The Same numbers can be written in the two boxes, The sum of these two numbers should be equal tol according to the first law

Then $a^1 = a^{\boxed{\frac{1}{2}}} \times a^{\boxed{\frac{1}{2}}}$

$a = a^{\boxed{\frac{1}{2}}} + a^{\boxed{\frac{1}{2}}}$

We can also write

$a^1 = a^{\boxed{\frac{1}{3}}} \times a^{\boxed{\frac{2}{3}}}$

$x = x^{\Box} \times x^{\Box} \times x^{\Box}$

Similarly, can you write the same number in all boxes above so that the above expression is true?

$x = x^{\boxed{\frac{1}{3}}} \times x^{\boxed{\frac{1}{3}}} \times x^{\boxed{\frac{1}{3}}} = x^{\boxed{\frac{1}{7}}} \times x^{\boxed{\frac{4}{7}}} \times x^{\boxed{\frac{2}{7}}}$

Activity 2.1

Write suitable fractions in the boxes as above.

(a) $5 = 5^{\Box} \times 5^{\Box} = 5^{\Box} \times 5^{\Box} \times 5^{\Box}$

(b) $2 = 2^{\Box} \times 2^{\Box} = 2^{\Box} \times 2^{\Box} \times 2^{\Box}$

(c) $p = p^{\Box} \times p^{\Box} = p^{\Box} \times p^{\Box} \times p^{\Box}$

(d) $q = q^{\Box} \times q^{\Box} = q^{\Box} \times q^{\Box} \times q^{\Box}$

2.2 Converting negative indices to positive indices

Simplify $\dfrac{a^2}{a^5}$

We can expand $\dfrac{a^2}{a^5}$ as $\dfrac{a \times a}{a \times a \times a \times a \times a}$

Hence, $\dfrac{a^2}{a^5} = \dfrac{\cancel{a}^1 \times \cancel{a}^1}{\cancel{a} \times \cancel{a} \times a \times a \times a} = \dfrac{1}{a^3}$

Also we get $\dfrac{a^2}{a^5} = a^{2-5} = a^{-3}$ (simplifying using the laws of indices)

Therefore $\dfrac{a^2}{a^5} = \dfrac{1}{a^3} = a^{-3}$

Thus, the power with a negative index a^{-3} can be written with a

positive index as $\dfrac{1}{a^3}$

When we take the reciprocal of a power, then the sign of its index changes from positive to negative and negative to positive.

Accordingly

$$\boxed{a^{-m} = \frac{1}{a^m}} \qquad \boxed{\frac{1}{a^{-m}} = a^m}$$

Example 1

Evaluate $\sqrt{25^3}$

$$\sqrt{25^3} = \left(25^3\right)^{\frac{1}{2}} = \left(25^{\frac{1}{2}}\right)^3 = \left(5^{2 \times \frac{1}{2}}\right)^3 = 5^3 = \underline{\underline{125}}$$

Example 2 Evaluate $(25)^{0.5}$

$$(25)^{0.5} = (25)^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$$

Example 3 Simplify $8^{\frac{1}{3}}, (16p^4)^{\frac{1}{4}}, (0.027)^{\frac{1}{3}}$

- $$8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = \underline{\underline{2}}$$
- $$(16p^4)^{\frac{1}{4}} = (2^4p^4)^{\frac{1}{4}} = \{(2p)^4\}^{\frac{1}{4}} = (2p)^{4\times\frac{1}{4}} = \underline{\underline{2p}}$$
- $$(0.027)^{\frac{1}{3}} = \left(\frac{27}{1000}\right)^{\frac{1}{3}} = \left(\frac{3^3}{10^3}\right)^{\frac{1}{3}} = \left\{\left(\frac{3}{10}\right)^3\right\}^{\frac{1}{3}} = \frac{3}{10} = \underline{\underline{0.3}}$$

Example 4 Simiplify $\sqrt[3]{a^2} \times \sqrt{a^3}$

$$\sqrt[3]{a^2} \times \sqrt{a^3} = (a^2)^{\frac{1}{3}} \times (a^3)^{\frac{1}{2}} = a^{\frac{2}{3}} \times a^{\frac{3}{2}} = a^{\frac{2}{3}+\frac{3}{2}} = \underline{\underline{a^{\frac{13}{6}}}}$$

Example 5 Write $\sqrt[3]{x^{-2}}$ with a positive index.

$$(x^{-2})^{\frac{1}{3}} = x^{-2\times\frac{1}{3}} = x^{-\frac{2}{3}} = \underline{\underline{\frac{1}{x^{\frac{2}{3}}}}}$$

Example 6 Evaluate $81^{-1\frac{1}{4}}$

$$81^{-1\frac{1}{4}} = 81^{-\frac{5}{4}}$$

$$= \frac{1}{81^{\frac{5}{4}}}$$

$$= \frac{1}{(3^4)^{\frac{5}{4}}}$$

$$= \frac{1}{3^{4\times\frac{5}{4}}}$$

$$= \frac{1}{3^5} = \underline{\underline{\frac{1}{243}}}$$

$$\boxed{(81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3}$$

Exercise 2.2

(1) Simplify

(i) $(36)^{\frac{1}{2}}$ (ii) $(64)^{\frac{1}{3}}$ (iii) $(196)^{\frac{1}{2}}$ (iv) $(4p^2q^2)^{\frac{1}{2}}$

(v) $(0.125)^{\frac{1}{3}}$ (vi) $\left(6\frac{1}{4}\right)^{\frac{1}{2}}$ (vii) $(256x^4)^{\frac{1}{2}}$ (viii) $12^{\frac{1}{2}} \times 3^{\frac{1}{2}}$

(ix) $\left(a^2\right)^3 \times \sqrt{a^4}$ (x) $\left(ab^2\right)^{\frac{1}{2}} \times a^{\frac{3}{2}}$

(2) Simplify

(i) $(64)^{\frac{2}{3}}$ (ii) $(729)^{\frac{2}{3}}$ (iii) $\left(\frac{64}{343}\right)^{\frac{2}{3}}$ (iv) $(0.01)^{\frac{1}{2}}$

(v) $(256)^{0.25}$ (vi) $(1000)^{\frac{-1}{3}}$ (vii) $(16)^{\frac{-3}{4}}$ (viii) $(243)^{\frac{-3}{5}}$

(ix) $(0.0001)^{-1\frac{1}{4}}$

(3) Find the value of each expression when $x = 8$ and $y = 16$

(i) $x^{\frac{1}{3}} \times y^{\frac{1}{2}}$ (ii) $2x^{\frac{1}{3}} \times y^{\frac{1}{2}}$ (iii) $\left(x^{\frac{1}{2}} y^{\frac{1}{3}}\right)^0$

(iv) $(x^{\frac{1}{3}} y^{\frac{1}{2}})^0$ (v) $\left(\frac{y}{x}\right)^2$

(4) Simplify and give the answers with positive indices.

(i) $10x^{-1} \times y^3 \times xy$ (ii) $(2a)^{-2} \times 8a^4$ (iii) $(2x)^2 \times \left(\frac{1}{128}\right)^{\frac{1}{7}}$

(iv) $\frac{\left(2a^{-2}\right)^3 \times \left(3a^{-4}\right)^2}{12a^{-3} \times 2a^{-2}}$ (v) $\left\{\left(3x^{-1}\right)^{-2}\right\}^{-1}$

(5) Simplify,

(i) $\frac{9a^{\frac{4}{3}} \times a^{\frac{-1}{2}}}{2a^{1\frac{1}{2}} \times 3a^{\frac{1}{3}}}$ (ii) $x^{\frac{2}{3}}\left(\left(x^{\frac{2}{3}} \div x^{\frac{1}{3}}\right) \div x\right)$ (iii) $\frac{\sqrt{x^3} \times \sqrt[3]{y^2}}{\sqrt[6]{y^{-2} \times \sqrt[4]{x^6}}}$

2.3 Solving equations with indices

Example 7 Solve $3^x = 81$,

$$3^x = 81$$

When expressing the number 81 as a power of 3, we get

$$3^x = 3^4$$

Since the bases of both powers are equal, the indices should also be equal,

$$\therefore x = 4$$

Example 8

Solve $16^{(x+1)} = 32$

$$16^{(x+1)} = 32$$

Write 16 and 32 as powers of the same base. 2 is an appropriate base for this

$$\left(2^4\right)^{(x+1)} = 2^5$$

$$2^{4x+4} = 2^5 \quad \text{(According to the law of indices)}$$

Since the bases are equal, the indices should also be equal.

$$\therefore \quad 4x + 4 = 5$$

$$4x + 4 - 4 = 5 - 4$$

$$4x = 1$$

$$x = \frac{1}{4}$$

Exercise 2.3

Find the value of x

(1) $5^x = 125$ (2) $27 = 3^{-x}$ (3) $7^{(x-1)} = 49$ (4) $4^x = \frac{1}{4}$

(5) $8^{(x+1)} \times 2^{(x-1)} = 32$ (6) $\left(4^2\right)^x \times 8^x = 128$ (7) $8^{(2x-1)} = \frac{1}{64}$

2.4 Laws of Logarithms

$$\boxed{\begin{array}{ll}\text{(i)} & \log_a mn = \log_a m + \log_a n \\ \text{(ii)} & \log_a \frac{m}{n} = \log_a m - \log_a n\end{array}}$$

You have already studied the above laws of logarithms. Let us see how these laws can be applied to powers and roots

Consider the expression $\log_a 2^3$ .This can be expanded as $\log_a (2\times2\times2)$.

According to the above laws,

$$\log_a 2^3 = \log_a 2\times2\times2 = \log_a 2 + \log_a 2 + \log_a 2 = 3\log_a 2$$

$$\therefore \log_a 2^3 = 3\log_a 2$$

Now consider $\log_a x^2$. This can be expanded as $\log_a x\times x$

$$\therefore \log_a x^2 = \log_a x \times x = \log_a x + \log_a x = 2\log_a x$$

$$\log_a x^2 = 2\log_a x$$

Similarly, $\log_a (x)^{\frac{1}{2}} = \frac{1}{2}\log_a x$

Now, we have another result for logarithms $\boxed{\log_a m^r = r\log_a m}$

where r is a rational number

Example 9

Find the value of $\log_7 343$

Since $7^3 = 343$

$$\log_7 343 = \log_7 7^3$$

$$= 3\log_7 7 \quad (\text{where } \log_7 7 = 1\ )$$

$$= 3$$

Example 10

Find the value of $lg0.001$

$$0.001 = \frac{1}{1000}$$

$$= \frac{1}{10^3} = 10^{-3}$$

Taking the logarithm of both sides, we have

$$lg0.001 = lg10^{-3} = -3lg10 = \underline{\underline{-3}}$$

> $log_{10}a$
> is written as $lg\, a$

> Here,
> $log_{10}10 = lg\ 10 = 1$

Example 11

Evaluate $lg(100)^{\frac{1}{2}}$

$$100^{\frac{1}{2}} = (10^2)^{\frac{1}{2}} = 10^{2\times\frac{1}{2}} = 10$$

$$\therefore lg100^{\frac{1}{2}} = lg10$$

$$lg100^{\frac{1}{2}} = \underline{\underline{1}}$$

Exercise 2.4

Evaluate

1. $log_3\ 27$
2. $log_2\ 256$
3. $log_4\ 256$
4. $log_2\ 8^3$
5. $lg\ 100^5$
6. $lg\ (1000)^{\frac{1}{2}}$
7. $log_2\ 64^{\frac{1}{2}}$
8. $log_3 81^{\frac{1}{4}}$

> x and y stand for two whole numbers.
> $log_3 x = log_2 y,\ x < 50$. Find all values that x and y can take . Compare your results with a friend of yours.

2.5 Finding the value of a logarithmic expression.

Example 12

Simplify $log_a 5 + 2\ log_a 3$

$= log_a 5 + log_a 3^2$

$= log_a (5 \times 3^2)$

$= \underline{\underline{log_a 45}}$

Example 13

Simplify $log_a 6 - log_a 3$

$log_a 6 - log_a 3 = log_a \frac{6}{3}$

$= \underline{\underline{log_a 2}}$

Example14

Evaluate $2lg5 + lg\,4$

$$\begin{aligned} 2lg5 + lg\,4 &= lg\,5^2 + lg4 \\ &= lg\,(5^2 \times 4) \\ &= lg\,100 \\ &= lg10^2 \\ &= \underline{\underline{2}} \end{aligned}$$

Example 15

Evaluate $\frac{1}{2} lg\ 25 + \frac{1}{2} lg\ \ 400$

$$\begin{aligned} \frac{1}{2} lg\ 25 + \frac{1}{2} lg\ 400 &= lg(25)^{\frac{1}{2}} + lg(400)^{\frac{1}{2}} \\ &= lg\left\{(25)^{\frac{1}{2}} \text{x } (400)^{\frac{1}{2}}\right\} \\ &= lg\left\{(5^2)^{\frac{1}{2}} \text{x} (20^2)^{\frac{1}{2}}\right\} \\ &= lg\ (5 \times 20) \\ &= lg\ 100 \\ &= lg\ 10^2 \\ &= \underline{\underline{2}} \end{aligned}$$

Exercise 2.5:

Find the value of each expression without using the logarithmic tables

(1) $\lg 5 + \lg 2$

(2) $lg\, 5 + lg\, 20$

(3) $lg\, 2000 - lg\, 200$

(4) $lg 16 - lg 4 + 2\, lg\, 5$

(5) $log_2 16 + 2\, log_2 8$

(6) $\frac{1}{2} log_3 81 + log_3 9$

(7) $2\, log_2 8 - \frac{1}{2} log_2 16$

(8) $lg\, 50 + 3\, lg\, 2 - 2\, lg\, 2$

(9) $\frac{1}{2} lg 10000 - 3 lg 10$

(10) $2\, lg 10 + \frac{1}{3} lg 1000 - 3$

(11) $\frac{1}{2}(lg 100 + lg\, 64 - lg\, 16 + 2\, lg 5)$

(12) $2(lg 20 - lg 2) + 1$

2.6 Solving logarithmic equations

Example 16

Solve $log_a 5 + log_a 2 = log_a x$

$$log_a 5 + log_a 2 = log_a x$$

$$log_a (5 \times 2) = log_a x$$

$$log_a 10 = log_a x$$

$$10 = x$$

$$\underline{\underline{x = 10}}$$

Example 17

Solve $log_a 4 - log_a 8 = log_a x$

$$log_a 4 - log_a 8 = log_a x$$
$$log_a \frac{4}{8} = log_a x$$
$$\frac{1}{2} = x$$
$$\underline{\underline{x = \frac{1}{2}}}$$

Example 18

Solve $lg\,20 + lg\,5 = 2\,lg\,x$

$$lg\,20 + lg\,5 = 2\,lg\,x$$
$$lg\,(20 \times 5) = lg\,x^2$$
$$lg\,100 = lg\,x^2$$
$$100 = x^2$$
$$10 = x \text{ or } -10 = x$$
$$\underline{\underline{x = 10}} \text{ (since } x > o)$$

Example 19

Solve $\frac{1}{2} log_a x = log_a 12 - log_a 3$

$$\frac{1}{2} log_a x = log_a 12 - log_a 3$$
$$log_a x^{\frac{1}{2}} = log_a \frac{12}{3}$$
$$x^{\frac{1}{2}} = 4$$
$$(x^{\frac{1}{2}})^2 = 4^2$$
$$x = 4^2 \text{ or } -10 = x$$
$$\underline{\underline{x = 16}} \text{ (since } x > o)$$

Example 20

Solve $2\ log_b\ 3 + 3\ log_b\ 2 - log_b\ 72 = \frac{1}{2}\ log_b x$

$$2\ log_b\ 3 + 3\ log_b\ 2 - log_b\ 72 = \frac{1}{2}\ log_b x$$

$$log_b 3^2 + log_b 2^3 - log_b 72 = log_b x^{\frac{1}{2}}$$

$$log_b \left\{\frac{3^2 \times 2^3}{72}\right\} = log_b (x^{\frac{1}{2}})$$

$$\frac{9 \times 8}{72} = x^{\frac{1}{2}}$$

$$1^2 = (x^{\frac{1}{2}})^2$$

$$1 = x^1$$

$$\underline{\underline{x = 1}}$$

Exercise 2.6

Solve the following equations

(1) $log_5\ 125 = x$

(2) $2\ log_a x = log_a 49$

(3) $log_a 25 + 2\ log_a x = 2\ log_a 50$

(4) $2\ log_a 4 - log_a 8 + log_a 50 = 2\ log_a x$

(5) $2\ log_a\ x = 4\ log_a 3$

(6) $log_a x = \frac{1}{2}\ log_a 144$

(7) $\frac{2}{3}\ log_a\ 8 = \frac{1}{2}\ log_a x$

(8) $log_a\ x = log_a\ (2x - 3)$