01 Real Numbers

Archimedes
212-287 B.C.
Archimedes, a mathematician and a Greek philosopher who introduced the value of π as $3\frac{10}{71} < \pi < 3\frac{1}{7}$. He used the mean value as the value of π

By studying this lesson you will acquire knowledge on the following :

- Identifying types of numbers (Counting numbers, integers, rational numbers, irrational numbers and real numbers)
- Identifying surds and entire surds
- Converting fractions with irrational denominator to fractions with rational denominator

1.1 Real Numbers

Children : Ayubowan Sir

Teacher : Ayubowan children,wish you a happy new year and a good day.

Teacher : You know about today's lesson. I informed you earlier to be prepared with facts needed for the lesson. I suppose that you are ready. Each group must present the facts they collected to the class as you have prepared. Nilupuli, present the facts that you have collected.

Nilupuli : The numbers that we first learnt about were the **counting numbers.** They are so called as they are the numbers used in counting.They are 1,2,3,4...Zero is not included in the set of counting numbers. When zero is included in the set of counting numbers,then it is known as the set of **whole numbers.** The set of whole numbers is symbolised by N i.e; N = { 0, 1, 2, . . .}. Then we learnt about **odd numbers** and **even numbers.** The numbers that are exactly divisible by 2 are called even numbers and the numbers which give a remainder of one when divided by 2 are called odd numbers. All these sets of numbers are infinite.

Nisam : After that, we learnt about number patterns. One of the first patterns we were taught was the pattern of Triangular numbers. The numbers 1,3,6,10,15.... belong to the pattern of triangular numbers. The numbers in the pattern 1, 4,9,16,25.... are called square numbers. We call them square numbers because they form a pattern of squares.

Nilupuli : We also learnt about prime numbers and composite numbers.
When an integer greater than 1 has exactly two factors, 1 and itself, it is called a **prime number**. For example, there fore, 2 and 5 are prime numbers. 1 is the only factor of 1. Therefore 1 is not a prime number.

- Factors of 2 are 1 and 2
- Factors of 5 are 1 and 5

Let's find out the details about composite numbers. A **composite number** is any integer greater than 1 that has more than two factors. For example, 6 is a composite number since it has four factors 1,2,3 and 6.

Yasangi : Fractions are another type of numbers we have learnt about.
Fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{17}, \frac{1}{25}$ with numerator 1 are called **Unit fractions.** Fractions such as $\frac{2}{5}, \frac{3}{7}, \frac{5}{9}, \frac{2}{12}$ where the numerator is greater than 1 and less than the denominator are called **proper fractions**. Fractions such as $\frac{11}{6}, \frac{7}{5}, \frac{8}{8}, \frac{17}{9}, \frac{15}{15}$ with numerator equal or greater than the denominator are called **improper fractions.** The numbers represented by a whole number and a fraction such as $1\frac{2}{3}, 4\frac{3}{5}, 12\frac{1}{8}$ are called **mixed numbers.**

Nuwan : On a number line, positive whole numbers written in ascending order from 0 to the right hand side are called **positive integers** (eg: +1,+2,+3 ......) and the numbers in descending order to the left hand side of 0 are called **negative integers.** The set of positive integers and negative integers including zero is named as the **integers**. The set of integers is denoted by $\mathbb{Z}$.

Fractions can be written as decimals.

When $\frac{3}{4}$ is written as a decimal it is 0.75 and $\frac{9}{8}$ as a decimal is 1.125.

But a fraction such as $\frac{2}{7}$ will give a decimal 0.285714285714..... where a row of numbers recur. Such decimals are called **recurring decimals.** When $\frac{1}{4}$ is written as a decimal it is 0.25 and when $\frac{3}{8}$ is written as a decimal it is 0.375. Such decimals which end up without repeating are called **finite decimals.** **Recurring decimals** and **finite decimals** can be written as a ratio or a fraction.

Teacher : Good. Now I will teach you more about numbers. The set of integers is denoted by $\mathbb{Z}$.

$$\mathbb{Z}^+ = \{+1,+2,+3,......\}$$
$$\mathbb{Z}^- = \{..........-3,-2,-1\}$$
$$\mathbb{Z} = \{..........-3,-2,-1,0,+1,+2,+3.......\}$$

Any number that can be expressed in the form of $\frac{p}{q}$,where p and q are integers and $q \neq 0$, is called a rational number.

We denote the set of rational numbers by $\mathbb{Q}$

Then $\mathbb{Q} = \{ \ x : x = \frac{p}{q} ; \ p,q \in \mathbb{Z}, q \neq 0 \ \}$

Also the set of all integers $\mathbb{Z}$ is a subset of the set of all rational numbers $\mathbb{Q}$. That is, $\mathbb{Z} \subset \mathbb{Q}$. A rational number can always be expressed as a finite decimal or a recurring decimal.

The numbers such as 2, $-8, \frac{3}{5}, 0.4, -\frac{7}{8}, \sqrt{25}, 3\sqrt{9}, 6\sqrt{64}$ are all rational numbers.But numbers such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{33}$ cannot be expressed as a ratio of two integers. On the other hand, $\sqrt{3}$ as a decimal is, 1.732050808... . It doesn't have any pattern.

$\sqrt{5}$ = 2.236067972..... and $\sqrt{33}$ = 5.744562647.... are two

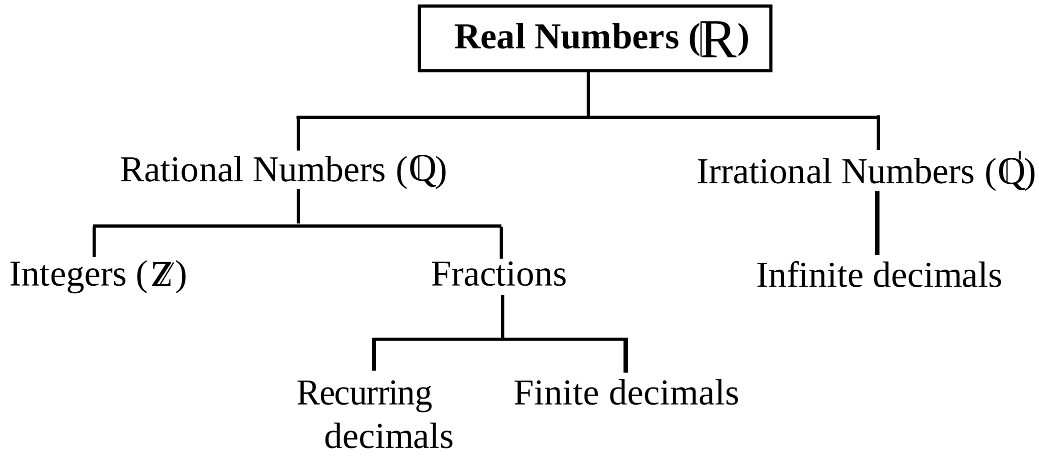
other examples. These are not recurring decimals either. How can we define these numbers? The numbers that cannot be expressed as a ratio of two intergers are called **irrational numbers.** Irrational numbers cannot be written as finite or recurring decimals.The set including both irrational numbers and rational numbers is known as the set of all **Real Numbers.** The set of real numbers is denote by ℝ. The set of all real numbers can be expressed as the union of the set of rational numbers and the set of irrational numbers. Let us find out whether the value of π is rational or irrational.We learnt that the value of π is the ratio of the circumference of a circle and its diameter.

We used $\frac{22}{7}$ as an approximate value for π.

$\frac{22}{7}$ = 3.142857..... is a recurring decimal. But π = 3 .141592653589793 23846 ... is neither a recurring decimal nor a finite decimal. Therefore, π belongs to the set of irrational numbers.

We use $\frac{22}{7}$ instead of π in calculations.

All the sets of numbers we learnt can be shown by the chart given below.



We have $\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}}$

Exercise 1 .1

(1) Write the following rational numbers as decimals. State whether they are finite decimals or recurring decimals.

(i) $\frac{3}{4}$ (ii) $\frac{5}{5}$ (iii) $\frac{3}{7}$ (iv) $\frac{5}{9}$ (v) $\frac{5}{11}$ (vi) $\frac{7}{8}$ (vii) $\frac{6}{5}$ (viii) $\frac{3}{5}$

(2) Choose the rational numbers and irrational numbers from among the real numbers given below.

(i) $\sqrt{16}$ (ii) $\sqrt{9}$ (iii) $\sqrt{7}$ (iv) 3.1010010001... (v) 5.136587321...

(vi) $\sqrt{17}$ (vii) $\sqrt{36}$ (viii) 2.12345678 (ix) 0.375 (x) $\sqrt{12}$

1.2 Surds

Numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}, 2\sqrt{2}, 7\sqrt{5}, \sqrt{10}, \sqrt{17}$ are named as irrational numbers. From these, the irrational numbers such as

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{10}, \sqrt{17}$ **,** $2\sqrt{2}$**,** $7\sqrt{5}$ are known as **surds.**

The exact value of $\sqrt{25}$ is 5 and the exact value of $\sqrt{16}$ is 4. That is, the values of $\sqrt{25}$ and $\sqrt{16}$ can be determined exactly. Therefore the numbers such as $\sqrt{25}$ and $\sqrt{16}$ are not surds. Surds of the form $\sqrt{32}$, $\sqrt{405}$ are called **entire surds.**

Example 1

Express the entire surd $\sqrt{32}$ as a surd

method 1 $\sqrt{32} = \sqrt{16 \times 2}$

It should be factorized such that one factor is a prime number while the other factor has an exact square root.

Then write it as two parts $= \sqrt{16} \times \sqrt{2}$

Because $\sqrt{16} = 4$,$\sqrt{32}$ can be written as $4 \times \sqrt{2}$ or $4\sqrt{2}$.

$4\sqrt{2}$, is a surd.

method 2

Express the entire surd $\sqrt{32}$ as a surd

2	32
2	16
2	8
2	4
2	2
	1

$$\begin{aligned}\sqrt{32} &= \sqrt{2 \times 2 \times 2 \times 2 \times 2} \\ &= \sqrt{2^2 \times 2^2 \times 2} \\ &= 2 \times 2\sqrt{2} \\ &= \underline{\underline{4\sqrt{2}}}\end{aligned}$$

Example 2

Express the entire surd $\sqrt{405}$ as a surd.

$$\sqrt{405} = \sqrt{81 \times 5}$$
$$= \sqrt{81} \times \sqrt{5}$$
$$= \underline{\underline{9\sqrt{5}}}$$

$\sqrt{405}, \sqrt{32}, \sqrt{20}, \sqrt{40},$ are some **entire surds.**
A surd can be converted to an entire surd too.

Example 3

Convert the surd $3\sqrt{5}$ to an entire surd.

If we write it as $\sqrt{\left(3\sqrt{5}\right)^2}$ we get $3\sqrt{5}$ back again.

Therefore we can write it as $\sqrt{\left(3\sqrt{5}\right)^2} = \sqrt{9 \times \sqrt{5} \times \sqrt{5}}$

$$= \sqrt{9 \times 5} \quad \left(\sqrt{5} \times \sqrt{5} = 5\right)$$

$= \underline{\underline{\sqrt{45}}}$, and this is an entire surd

Example 4

Convert the surd $2\sqrt{7}$ to an entire surd.

$$\sqrt{\left(2\sqrt{7}\right)^2} = \sqrt{2^2 \times \sqrt{7} \times \sqrt{7}}$$
$$= \sqrt{4 \times 7}$$

$= \underline{\underline{\sqrt{28}}}$, and this is an entire surd.

1.3 Addition and subtraction of surds

Example 5

Simplify .

$$\sqrt{2}+6\sqrt{5}+4\sqrt{2}-3\sqrt{5}-3\sqrt{2}.$$

Recall how you added and subtracted algebraic expressions such as $2x+3y+5x-y$. Accordingly, let us simplify expressions with surds.

$4\sqrt{2}$ represents 4 times $\sqrt{2}$

Therefore, $\sqrt{2}+4\sqrt{2}-3\sqrt{2}=2\sqrt{2}$

$$6\sqrt{5}-3\sqrt{5}=3\sqrt{5}$$

$$\therefore \sqrt{2}+4\sqrt{2}-3\sqrt{2}+6\sqrt{5}-3\sqrt{5}=\underline{\underline{2\sqrt{2}+3\sqrt{5}}}$$

There may be entire surds in certain additions or subtractions. In such problems, the simplification must be done after converting the entire surds into surds. (Convert all any entire surds into surds before you do any simplification.)

Example 6

Simplify $\quad 4\sqrt{63}-5\sqrt{7}-8\sqrt{28}$

$$4\sqrt{63}=4\sqrt{9\times 7}$$

$$4\sqrt{63}=4\times 3\sqrt{7}$$

$$=12\sqrt{7}$$

$$8\sqrt{28}=8\sqrt{4\times 7}$$

$$=8\times 2\sqrt{7}$$

$$=16\sqrt{7}$$

Hence $\quad 4\sqrt{63}-5\sqrt{7}-8\sqrt{28}=12\sqrt{7}-5\sqrt{7}-16\sqrt{7}$

$$=\underline{\underline{-9\sqrt{7}}}$$

1.4 Multiplication and division of surds

Example 7

Simplyfy $4\sqrt{3}\times5\sqrt{2}$.

Multiply rational numbers and irrational numbers separately.

$$\therefore 4\sqrt{3}\times5\sqrt{2}=4\times5\times\sqrt{3}\times\sqrt{2}=\underline{\underline{20\sqrt{6}}} \quad \left(\sqrt{3}\times\sqrt{2}=\sqrt{3\times2}=\sqrt{6}\right)$$

Example 8

Simplyfy $2\sqrt{28}\div3\sqrt{7}$

$2\sqrt{28}$ is an entire surd. It can be written as $2\sqrt{4\times7}=2\times2\sqrt{7}=4\sqrt{7}$

$$\frac{2\sqrt{28}}{3\sqrt{7}}=\frac{2\times\sqrt{4\times7}}{3\sqrt{7}}=\frac{4\sqrt{7}}{3\sqrt{7}}=\frac{4}{\underline{\underline{3}}}$$

Example 9

$5\sqrt{12}\div4\sqrt{2}$.

$5\sqrt{12}$ is an entire surd.

$$5\sqrt{12}=5\sqrt{4\times3}=5\times2\sqrt{3}=10\sqrt{3}$$

$$\therefore \quad \frac{5\sqrt{12}}{4\sqrt{2}}=\frac{10\sqrt{3}}{4\sqrt{2}}=\frac{5\sqrt{3}}{2\sqrt{2}}=\frac{5}{2}\sqrt{\frac{3}{2}}$$

Exercise 1.2

(1) Write the following entire surds as surds.

(i) $\sqrt{20}$ (ii) $\sqrt{48}$ (iii) $\sqrt{72}$ (iv) $\sqrt{28}$ (v) $\sqrt{80}$ (vi) $\sqrt{45}$ (vii) $\sqrt{98}$

(viii) $\sqrt{147}$

(2) Convert the following surds in to entire surds.

(i) $2\sqrt{3}$ (ii) $3\sqrt{5}$ (iii) $4\sqrt{7}$ (iv) $5\sqrt{2}$ (v) $6\sqrt{11}$

(3) Simplify

(i) $\sqrt{2}+\sqrt{3}+5\sqrt{2}+3\sqrt{3}-2\sqrt{2}$

(ii) $\sqrt{5}+2\sqrt{7}+2\sqrt{5}-3\sqrt{7}$

(iii) $4\sqrt{3}+5\sqrt{2}+3\sqrt{5}-3\sqrt{2}+3\sqrt{5}-2\sqrt{3}$

(iv) $6\sqrt{11}+3\sqrt{7}-2\sqrt{11}-5\sqrt{7}+4\sqrt{7}$

(v) $8\sqrt{3}+7\sqrt{7}-2\sqrt{3}+3\sqrt{7}-3\sqrt{7}$

(4) Simplify.

(i) $3\sqrt{2}\times 2\sqrt{3}$ (ii) $5\sqrt{11}\times 3\sqrt{7}$ (iii) $\sqrt{5}\times 3\sqrt{3}$

(iv) $4\sqrt{7}\div 2\sqrt{14}$ (v) $6\sqrt{27}\div 3\sqrt{3}$ (vi) $\sqrt{48}\div 5\sqrt{3}$

(5) Simplify.

(i) $2\sqrt{27}-3\sqrt{3}+4\sqrt{7}+3\sqrt{28}$

(ii) $3\sqrt{63}-2\sqrt{7}-3\sqrt{27}+3\sqrt{3}$

(iii) $2\sqrt{128}+3\sqrt{50}-2\sqrt{162}+4\sqrt{2}$

(iv) $\sqrt{99}-2\sqrt{44}+6\sqrt{11}$

Example 10

Taking $\sqrt{2} = 1.414$, (i) find the value of $\frac{\sqrt{2}}{2}$

(ii) find the value of $\frac{2}{\sqrt{2}}$

(i) Taking $\sqrt{2} = 1.414$,we can easily simplify $\frac{\sqrt{2}}{2}$ as $\frac{1.414}{2} = 0.707.$

(ii) But it is not easy to simplify $\frac{2}{\sqrt{2}} = \frac{2}{1.414}$ as earlier.

It can be simplified as,

$$\frac{2}{1.414} = \frac{2000}{1414} = 1414\overline{)2000} = 1.414$$

In (i) above, the denominator is rational and in (II) the denominator is irrational.

* It is difficult to simplify a fraction when the denominator of the fraction is irrational.

Let us find how we can convert an irrational denominator to a rational denominator.

Consider $\sqrt{x} \times \sqrt{x}$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt{x} \times \sqrt{x} = x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\left(\frac{1}{2} + \frac{1}{2}\right)} = x$$

$$\therefore \sqrt{x} \times \sqrt{x} = x.$$

Similarly, $\sqrt{2} \times \sqrt{2} = 2$ and $\sqrt{a} \times \sqrt{a} = a$. Therefore we can remove the square root sign using indices. This can be used to convert an irrational denominator of a fraction into a rational denominator. Let us convert the denominator of $\frac{2}{\sqrt{2}}$ into a rational denominator.

Multiplying both the numerator and the denominator by $\sqrt{2}$,
We have,

$\frac{2 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2} = 1.414$. We obtained the same result in (ii) above.

You would notice that this method is much easier than the first method.

To convert a fraction with an irrational denominator into a rational denominator, both the numerator and the denominator must be multiplied by the irrational denominator. This is called rationalising the denominator.

Then the denominator becomes rational and it is easy to simplify even though the numerator becomes irrational. Look at the examples given below.

Example 11 Rationalising the denominator of $\frac{a}{\sqrt{b}}$

$$\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \underline{\underline{\frac{a\sqrt{b}}{b}}}$$

Example 12 Rationalising the denominator of $\frac{3\sqrt{5}}{\sqrt{3}}$

$$\frac{3\sqrt{5}}{\sqrt{3}} = \frac{3\sqrt{5} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\cancel{3}\sqrt{15}}{\cancel{3}} = \underline{\underline{\sqrt{15}}}$$

Exercise1.3

(1) Rationalise the denominator of the following fractions.

(i) $\frac{2}{\sqrt{7}}$ (ii) $\frac{3\sqrt{2}}{\sqrt{3}}$ (iii) $\frac{5\sqrt{3}}{\sqrt{5}}$ (iv) $\frac{4\sqrt{7}}{2\sqrt{3}}$ (v) $\frac{1}{\sqrt{7}}$

(vi) $\frac{3}{\sqrt{5}}$ (vii) $\frac{3}{2\sqrt{5}}$ (viii) $\frac{3\sqrt{5}}{2\sqrt{7}}$ (ix) $\frac{2\sqrt{3}}{3\sqrt{2}}$ (x) $\frac{3\sqrt{3}}{2\sqrt{5}}$

(2) Assuming that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, and $\sqrt{5} = 2.236$, find the value of each fraction given below.

(i) $\frac{3}{\sqrt{2}}$ (ii) $\frac{5}{\sqrt{3}}$ (iii) $\frac{2\sqrt{3}}{3}$ (iv) $\frac{3\sqrt{5}}{2}$ (v) $\frac{3\sqrt{3}}{\sqrt{5}}$

1 Egyptians used $\frac{256}{81}$ as the value of π.

2 The mathematician Archimedes has shown that $\frac{22}{7}$ and $\frac{231}{71}$ are suitable approximate rational values for π.

3 In 430 - 501 AD a Chinese mathematician named Zu chongzhi showed that the value of π is approximately equal to $\frac{355}{113}$

4 In 1000 BC, Babylonians used $3\frac{1}{8}$ as the value of π

5 In 1914 AD, the Indian mathematician named Ramanujan showed that $\sqrt{\sqrt{\frac{2143}{22}}}$ $\left(= \left[\frac{2143}{22}\right]^{\frac{1}{4}}\right)$ can be used for π.