19 Logarithms I

By studying this lesson you will be able to

simplify numerical expressions using logarithms.

Indices

The result of multiplying 2 by itself four times is expressed as 2^4.

That is, $2 \times 2 \times 2 \times 2 = 2^4$.

Therefore, the value of 2^4 is 16.

Similarly, $3 \times 3 \times 3 = 3^3 = 27$.

Expressions such as 2^4 and 3^3 are called powers. The base of 2^4 is 2 while the index is 4. Do the following exercise to review the facts that you have learnt so far regarding indices.

Review Exercise

1. For each term in box A select the term in box B which is equal to it and join them together.

$a \times a$	5^{-1}
a^{-2}	$a \times a \times b \times b$
$5 \times 5 \times 5 \times 5 \times 5$	x
a	5^5
a^2b^2	$\frac{a}{b}$
5^1	a^2
$\frac{1}{5}$	$\frac{1}{a^2}$
$5 \times 5^\circ$	1
x°	a^1
$5^3 \times 5^2$	5
ab^{-1}	

2. Fill in the blanks.

(i) $\frac{1}{8}=\frac{1}{2^{\cdots}}=2^{\cdots}$ (ii) $\frac{1}{100}=\frac{1}{10^{\cdots}}=10^{\cdots}$ (iii) $\frac{1}{125}=\frac{1}{5^{\cdots}}=5^{\cdots}$

(iv) $\frac{1}{81}=\frac{1}{3^{\cdots}}=3^{\cdots}$ (v) $0.01=\frac{1}{\cdots\cdot}=\frac{1}{10^{\cdots\cdot}}=\cdots\cdot$ (vi) $0.001=\frac{1}{\cdots\cdot}=\frac{1}{\cdots\cdot}=\cdots\cdot$

3. Simplify each of the following expressions.

(i) $a^2\times a^3$ (ii) $x^5\times x$ (iii) $\frac{x^5\times x^7}{x^{11}}$

(iv) $\frac{a^3\times a^5}{a^2\times a^6}$ (v) $\frac{p^3\times p^{-1}}{p}$ (vi) $\frac{x^{\circ}\times x^5}{x}$

4. Simplify each of the following expressions and find its value.

(i) $2^2\times 2^3$ (ii) $\frac{3^7}{3^4}$ (iii) $\frac{3^2\times 3^8}{3^5}$

(iv) $\frac{5^3\times 5^0}{5}$ (v) $\frac{10^2\times 10^3}{10\times 10^4}$ (vi) $\frac{2^5\times 2^3}{2^6\times 2^2}$

19.1 Logarithms

Let us now consider how simplifications are facilitated by using the properties of indices. To do this, let us use the table of powers of 2 given below.

Power of 2	2°	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Value	1	2	4	8	16	32	64	128	256	512	1024

Let us consider how the value of $\frac{64\times 512}{128}$ is found using this table.

First, let us write these numbers as powers of the same base.

$$\frac{64\times 512}{128} = \frac{2^6\times 2^9}{2^7} \quad \text{(According to the table)}$$

$$= 2^{6+9-7} \quad \text{(Using the laws of indices)}$$

$$= 2^8$$

$$= \underline{\underline{256}} \quad \text{(According to the table)}$$

It can be seen that the above simplification was done easily and concisely by using the laws of indices. In the above example it was possible to write the numbers as powers of 2. Any expression containing the multiplication and division of numbers can easily be simplified using the logarithm tables. John Napier (1550 A.D. -1617 A.D.) a Scottish mathematician is bestowed with the honour of introducing the logarithm tables first. Briggs another mathematician who was a contemporary of Napier developed logarithms further. Although the use of the logarithm tables has reduced in recent years due to the widespread use of calculators, it is very important to learn the mathematical concepts related to logarithms.

Index form and logarithm form

We know that $2^3 = 8$. Here 8 is expressed as a power with base 2 and index 3. Such expressions are defined as " index form". This can also be expressed as the logarithm of 8 to base 2 is 3. This is written as $\log_2 8 = 3$, which is defined as the "logarithm form". It must be clear to you that the same statement is written in index form and in logarithm form.

Accordingly, since $2^3 = 8$, we also have $\log_2 8 = 3$.
Similarly, $\log_2 8 = 3$, means $2^3 = 8$.

Let us consider several other examples.
- Since $3^2 = 9$, the logarithm of 9 to base 3 is 2. That is, $\log_3 9 = 2$.
- Since $5^1 = 5$, the logarithm of 5 to base 5 is 1. That is, $\log_5 5 = 1$.
- Since $10^3 = 1000$, the logarithm of 1000 to base 10 is 3. That is, $\log_{10} 1000 = 3$.

In general, for a positive number a,

$$\boxed{\text{If } a^x = N, \text{ then } \log_a N = x} \qquad \boxed{\text{If } \log_a N = x, \text{ then } a^x = N}$$

$a^x = N$ is considered as the index form and $\log_a N = x$ is considered as the logarithm form. Here a and N take only positive values. (Since any power of a positive number is positive, in the above relationship, when a is positive, N is also positive.) Accordingly, when considering logarithms, the base always takes a positive value.

Let us now identify several properties of logarithms.

(i) For any base, the logarithm of the base value itself is 1.

That is, $\log_a a = 1$.
This is because $a^1 = a$.
For example, $\log_2 2 = 1$ and $\log_{10} 10 = 1$.

(ii) The logarithm of 1 to any base (other than 1) is 0.

That is, $\log_a 1 = 0$.
This is because $a^0 = 1$.
For example, $\log_2 1 = 0$ and $\log_{10} 1 = 0$.

We observe that so far we have obtained positive values for the logarithms. However, logarithms can also be negative. The logarithm of a number between 0 and 1 is always negative.
For example,

since $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, we obtain, $\log_2\left(\frac{1}{8}\right) = -3$,

since $0.01 = \frac{1}{100} = 10^{-2}$, we obtain, $\log_{10}\left(\frac{1}{100}\right) = -2$,

and since $0.5 = \frac{5}{10} = 2^{-1}$, we obtain, $\log_2(0.5) = -1$,

Now let us consider how equations involving logarithms are solved.

Example 1

Find the value represented by x in each of the following.

(i) $\log_2 64 = x$ (ii) $\log_x 81 = 4$ (iii) $\log_5 x = 2$

i) $\log_2 64 = x$

$2^x = 64$ (In index form)

$2^x = 2^6$

$\therefore \underline{\underline{x = 6}}$

(ii) $\log_x 81 = 4$

$x^4 = 81$

$x^4 = 3^4$

$x = \pm 3$

$x = +3$ or -3

Since the base of a logarithm cannot be negative

$\underline{\underline{x = +3}}$

(iii) $\log_5 x = 2$

$x = 5^2$

$\underline{\underline{x = 25}}$

Exercise 19.1

1. Write each of the following expressions in logarithm form.

(i) The logarithm of 32 to base 2 is 5.
(ii) The logartihm of 1000 to base 10 is 3.
(iii) The logarithm of x to base 2 is y.
(iv) The logarithm of q to base p is r.
(v) The logarithm of r to base q is p.

2. Express each of the following in index form.

(i) $\log_5 125 = 3$ (ii) $\log_{10} 100\,000 = 5$ (iii) $\log_a x = y$

(iv) $\log_p a = q$ (v) $\log_a 1 = 0$ (vi) $\log_m m = 1$

3. Express each of the following in logarithm form.

(i) $2^8 = 256$ (ii) $10^4 = 10000$ (iii) $7^3 = 343$

(iv) $20^2 = 400$ (v) $a^x = y$ (vi) $p^a = q$

4. Solve each of the following equations.

(i) $\log_3 243 = x$ (ii) $\log_{10} 100 = x$ (iii) $\log_6 216 = x$

(iv) $\log_x 25 = 2$ (v) $\log_x 64 = 6$ (vi) $\log_x 10 = 1$

(vii) $\log_3 x = 2$ (viii) $\log_{10} x = 4$ (ix) $\log_8 x = 2$

5. (i) Write 64 as a power of four different bases.

(ii) Find four distinct pairs of values for x and y such that $\log_x 64 = y$.

19.2 Laws of logarithms

Let us recall again how the value of 16×32 can be obtained by writing it in index form.

$$\begin{aligned} 16 \times 32 &= 2^4 \times 2^5 \\ &= 2^{4+5} \\ &= 2^9 \end{aligned}$$

Let us now consider $16 \times 32 = 2^{4+5}$.

Let us convert this to logarithm form.

$$16 \times 32 = 2^{4+5} \quad \text{(Index form)}$$

$$\therefore \log_2(16 \times 32) = 4 + 5 \quad \text{(Logarithm form)}$$

$$= \log_2 16 + \log_2 32 \quad (\text{Since } 4 = \log_2 16 \text{ and } 5 = \log_2 32)$$

Similarly, since $27 \times 81 = 3^3 \times 3^4 = 3^{3+4}$,

$$\log_3(27 \times 81) = 3 + 4.$$

$$= \log_3 27 + \log_3 81 \quad (\text{Since } 3 = \log_3 27 \text{ and } 4 = \log_3 81)$$

In the same manner we can write,

$$\log_{10}(10 \times 100) = \log_{10} 10 + \log_{10} 100 \text{ and}$$

$$\log_5(125 \times 25) = \log_5 125 + \log_5 25.$$

As seen above, when multiplying powers, an important fact about the behaviour of logarithms is highlighted. This is true in general for any product of powers and is expressed as follows.

$$\boxed{\log_a(mn) = \log_a m + \log_a n}$$

This statement can also be expressed as "the logarithm of a product is equal to the sum of the logarithms".

Such a formula exists for the logarithm of a quotient too. Let us investigate this now.

Let us consider the following example.

Let us recall how the value of 128 ÷ 16 is obtained by converting it into index form.

$$\frac{128}{16} = \frac{2^7}{2^4} \quad \text{(Representing as powers of 2)}$$

$$= 2^{7-4} \quad \text{(Applying the laws of indices)}$$

$$\therefore \log_2\left(\frac{128}{16}\right) = 7 - 4 \quad \text{(Writing in logarithm form)}$$

Now, since $128 = 2^7$, we obtain $7 = \log_2 128$,

and since $16 = 2^4$, we obtain $4 = \log_2 16$.

Accordingly, $\log_2\left(\frac{128}{16}\right) = 7 - 4 = \log_2 128 - \log_2 16$.

Similarly, $\log_5(125 \div 5) = \log_5 125 - \log_5 5$

and $\log_{10}\left(\frac{1000}{100}\right) = \log_{10} 1000 - \log_{10} 100$.

As seen above, when dividing powers, an important fact about the behaviour of logarithms is highlighted. This is true in general for any quotient of powers and is expressed as follows.

$$\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}$$

These properties are called "Laws of Logarithms."

Now let us learn how to solve problems using these laws of logarithms by considering the following examples.

Example 1

1. Find the value of each of the expressions given below.

   (i) $\log_4 32 + \log_4 2$ (ii) $\log_5 15 - \log_5 3$

(i) $\log_4 32 + \log_4 2 = \log_4(32 \times 2)$

$= \log_4 64$

$= \underline{\underline{3}} \qquad (64 = 4^3)$

(ii) $\log_5 15 - \log_5 3 = \log_5\left(\frac{15}{3}\right)$

$= \log_5 5$

$= \underline{\underline{1}}$

Example 2

Evaluate

$$\log_{10}25 + \log_{10}8 - \log_{10}2$$

$$\log_{10} 25 + \log_{10} 8 - \log_{10} 2 = \log_{10}\left(\frac{25 \times \cancel{8}^{4}}{\cancel{2}^{1}}\right)$$

$$= \log_{10} 100$$

$$= \underline{\underline{2}}$$

$$\boxed{\begin{aligned} \log_{10} 100 &= x \\ 10^x &= 100 \\ 10^x &= 10^2 \\ \therefore x &= 2 \end{aligned}}$$

Example 3

Express in terms of $\log_a 2$ and $\log_a 3$.

(i) $\log_a 6$

(ii) $\log_a 18$

(i) $6 = 2 \times 3$

$\log_a 6 = \log_a(2 \times 3)$

$= \underline{\underline{\log_a 2 + \log_a 3}}$

(ii) $18 = 2 \times 3 \times 3$

$\log_a 18 = \log_a(2 \times 3 \times 3)$

$= \log_a 2 + \log_a 3 + \log_a 3$

$= \underline{\underline{\log_a 2 + 2\log_a 3}}$

Example 4

Solve:

$$\log_a 5 + \log_a x = \log_a 3 + \log_a 10 - \log_a 2$$

$$\log_a 5 + \log_a x = \log_a 3 + \log_a 10 - \log_a 2$$

$$\Rightarrow \log_a (5 \times x) = \log_a\left(\frac{3 \times 10}{2}\right)$$

$$\therefore \quad 5x = \frac{3 \times \cancel{10}^{5}}{\cancel{2}_{1}}$$

$$\Rightarrow 5x = 15$$

$$\underline{\underline{x = 3}}$$

Now, do the following exercise by applying the laws of logarithms.

Exercise 19.2

1. Simplify and express the answer as a single logarithm.

(i) $\log_2 10 + \log_2 5$ (ii) $\log_3 8 + \log_3 5$ (iii) $\log_2 7 + \log_2 3 + \log_2 5$

(iv) $\log_6 20 - \log_6 4$ (v) $\log_a 10 - \log_a 2 - \log_a 5$ (vi) $\log_{10} 6 + \log_{10} 2 - \log_{10} 3$

2. Find the value of each of the following expressions.

(i) $\log_2 4 + \log_2 8$ (ii) $\log_3 27 - \log_3 3$

(iii) $\log_{10} 20 + \log_{10} 2 - \log_{10} 4$ (iv) $\log_2 80 - \log_2 15 + \log_2 12$

(v) $\log_{10} 20 + \log_{10} 10 - \log_{10} 2$ (vi) $\log_5 20 + \log_5 4 - \log_5 16$

3. Write the following expressions in terms of $\log_a 5$ and $\log_a 3$.

(i) $\log_a 15$ (ii) $\log\left(\frac{5}{3}\right)$ (iii) $\log_a\left(\frac{5}{3}\right)$

(iv) $\log_a 45$ (v) $\log_a 75$ (vi) $\log_a(225)$

4. Solve the following equations.

(i) $\log_2 5 + \log_2 3 = \log_2 x$ (ii) $\log_a 10 + \log_a x = \log_a 30$

(iii) $\log_3 20 + \log_3 x = \log_3 4 + \log_3 10$ (iv) $\log_a 15 - \log_a 3 = \log_a x$

(v) $\log_{10} 8 + \log_{10} x - \log_{10} 2 = \log_{10} 12$ (vi) $\log_5 24 - \log_5 4 = \log_5 2 + \log_5 x$

Summary

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a a = 1 \text{ and } \log_a 1 = 0 \ (a \neq 1)$$

Miscellaneous Exercise

1. Evaluate the following.

(i) $\log_3 27 + \log_2 8$ (ii) $\log_3 243 - \log_3 27$ (iii) $\log_2 16 \times \log_3 9$

(iv) $\dfrac{\log_{10} 10}{\log_2 32}$ (v) $\log_a 5 + \log_a 3 - \log_a 15$

2. If $\log_2 24 = x$, express $\log_2 48$ in terms of x.

3. Verify each of the following equations.

(i) $\log_a\left(\frac{9}{10}\right)+\log_a\left(\frac{25}{81}\right)=\log_a 5-\log_a 18$

(ii) $\log_5 1+\log_5 20-\log_5 8+\log_5 2=1$

(iii) $\log_{10} 2+\log_{10} 3-1=\log_{10} 0.6$

4. Evaluate.

(i) $\log_{10} 200+\log_{10} 300-\log_{10} 60$

(ii) $\log_{10}\left(\frac{12}{5}\right)+\log_{10}\left(\frac{25}{21}\right)-\log_{10}\left(\frac{2}{7}\right)$

5. Solve.

(i) $\log_{10} x-\log_{10} 2=\log_{10} 3-\log_{10} 4+1$

(ii) $\log_2 12-\log_2 3=\log_2 x+1$