15 Equations

By studying this lesson you will be able to

- construct and solve simple equations with algebraic fractions
- construct and solve simultaneous equations
- solve quadratic equations using factors

Solving simple equations

Let us work on the following exercise in order to revise the knowledge we have acquired on solving simple equations.

Review Exercise

1. Solve the following equations.

a. $2x+8=x+12$ **b.** $2(x-3)=4$ **c.** $5x-8=2(3-x)$

d. $2(y+3)=3(y-1)$ **e.** $4-5(3-p)=2(p-1)$ **f.** $\frac{x}{2}+1=3$

g. $5-\frac{x}{4}=1$ **h.** $3-\frac{2x}{5}=1$ **i.** $\frac{x}{3}+\frac{x}{4}=7$

j. $\frac{5x-2}{4}=2$ **k.** $\frac{(a-3)}{2}+1=4$ **l.** $\frac{(x+1)}{2}+\frac{(x-3)}{4}=\frac{1}{2}$

15.1 Solving simple equations described further:

Let us learn how to construct and solve an equation.

Did you observe that some of the equations in the above excercise contain fractions. The unknown term (x, y, p, a) was in the numerator of the fraction?. Now we are going to construct and solve equations with fractions of which the unknown term is in the denominator. Let us construct and solve such an equation.

Twelve is divided by two numbers of which the value of one number is double that of the other. The difference between the two quotients is 2. Find the two numbers.

Let us examine how this problem can be solved by the trial and error method.

Case I: Can the two numbers be 2 and 4?

$\frac{12}{2}=6$ and $\frac{12}{4}=3$, so the difference is $6-3=3$, and they do not work.

Case II: Can the two numbers be 6 and 12?

$\frac{12}{6}=2$ and $\frac{12}{12}=1$, so the difference is $2 - 1 = 1$, and they do not work.

Case III: Can the two numbers be 3 and 6?

$\frac{12}{3}=4$ and $\frac{12}{6}=2$, so the difference is $4 - 2 = 2$, and they work!

Thus, the problem can be solved using the method of trial and error. However, for some problems, the trial and error method is too long. Also, some problems cannot be solved by trial and error method. A suitable method of solving this type of problems is by using equations. Now, let us see how it can be solved by constructing an equation.

Suppose 12 is divided by a number x and its double $2x$.

Then the quotient of 12 divided by x is $\frac{12}{x}$.

The quotient of 12 divided by twice the x, i.e., $2x$, is $\frac{12}{2x}$.

Because the difference between the two quotients is 2, we get $\frac{12}{x} - \frac{12}{2x} = 2$.

The value of x obtained by solving this equation gives us the required number. Let us solve this equation. This is an equation having fractions with algebraic terms in the denominator. The first fraction contains x in the denominator. The second fraction has $2x$ in the denominator. Let us make the denominators of both fractions equal. The easiest way to do this is by replacing $\frac{12}{x}$ by the equivalent fraction $\frac{12\times 2}{x\times 2}$, which is $\frac{24}{2x}$.

Then,

$$\frac{24}{2x} - \frac{12}{2x} = 2$$

$$\therefore \frac{12}{2x} = 2$$

Multiplying both sides by $2x$

$$\frac{12}{2x} \times 2x = 2\times 2x$$

$$\text{i.e., } 12 = 4x$$

Dividing both sides by 4

$$\frac{12}{4} = \frac{4x}{4}$$

$$\therefore 3 = x, \text{ i.e., } x = 3$$

It follows that the two numbers are 3 and 6.

Note: This equation can be solved by writing the equation $\frac{12}{2x} = 2$ as $12 = 4x$ by cross multiplication too.

Example 1

Several friends shared 60 mangoes equally. One of them, Amal, sold three of his mangoes, and then he had only two mangoes left. How many friends shared the mangoes?

In reality, this problem can easily be solved mentally. Nevertheless, let us solve this equation as follows just to illustrate the construction and solving of equations.

Let x be the number of friends.

Then, the number of mangoes one person obtains $= \frac{60}{x}$

Number of mangoes Amal sold $= 3$

Then, the number of mangoes left $= \frac{60}{x} - 3$

Since he had only two left, we may write,

$$\frac{60}{x} - 3 = 2$$

Let us now solve this equation.

Let us add 3 to both sides.

$$\frac{60}{x} - 3 + 3 = 2 + 3$$

$$\therefore \frac{60}{x} = 5$$

Then, $5x = 60$

Hence, $x = 12$.

Therefore, 12 friends shared mangoes.

Observe how the following equations are solved.

Example 2

$$\frac{3}{a} + \frac{2}{a} = \frac{1}{2}$$

$$\frac{5}{a} = \frac{1}{2}$$

By cross multiplying,

$$\underline{\underline{a = 10}}$$

Example 3

$$\frac{3}{(x+2)} = \frac{1}{2}$$

By cross multiplying we obtain,

$$1 \times (x+2) = 2 \times 3$$

$$x + 2 = 6$$

$$\underline{\underline{x = 4}}$$

Example 4

$$\frac{2}{(x+5)} = \frac{3}{2(x-3)}$$

$$4(x-3) = 3(x+5)$$

$$4x-12 = 3x+15$$

$$4x-3x = 15+12$$

$$\underline{\underline{x = 27}}$$

Example 5

$$\frac{2}{(x-1)} - \frac{1}{2(x-1)} = \frac{3}{4}$$

$$\frac{4-1}{2(x-1)} = \frac{3}{4}$$

$$\frac{3}{2(x-1)} = \frac{3}{4}$$

$$3 \times 2\,(x-1) = 3 \times 4$$

$$\cancel{3}^{1} \times \cancel{2}^{1}(x-1) = \cancel{3}^{1} \times \cancel{4}^{2}$$

$$x-1=2$$

$$\underline{\underline{x=3}}$$

Exercise 15.1

1. A father and his sons equally shared an amount of Rs 270. Then the amount each person had was Rs 45. Taking the number of sons as x, construct an equation. Solve this equation and hence find the number of sons the father has.

2. When the same number was added both to the numerator and the denominator of the fraction $\frac{3}{5}$, the resulting fraction was equal to $\frac{9}{10}$. What number was added?

3. Solve the following equations.

a. $\frac{5}{m} + \frac{2}{m} = \frac{1}{2}$ **b.** $\frac{3}{5x} + \frac{1}{x} = 2$ **c.** $\frac{5}{6x} - \frac{2}{3x} = \frac{1}{6}$

d. $\frac{4}{5x} - \frac{1}{3x} = \frac{7}{30}$ **e.** $\frac{21}{4m+1} = 3$ **f.** $\frac{3}{x+2} = \frac{3}{7}$

g. $\frac{10}{a-3} = \frac{5}{8}$ **h.** $\frac{4}{x+1} = \frac{3}{x-2}$ **i.** $\frac{2}{x-3} = \frac{3}{x+8}$

j. $\frac{1}{a+1} + \frac{3}{a+1} = \frac{2}{3}$ **k.** $\frac{5}{x-2} + \frac{3}{x-2} = 2$ **l.** $\frac{5}{2(p+1)} + \frac{1}{p+1} = \frac{7}{8}$

m. $\frac{3}{x+2} - \frac{1}{3(x+2)} = \frac{8}{15}$ **n.** $\frac{1}{2x-3} + \frac{4}{x+3} = 0$ **o.** $\frac{15}{2(p+1)} - \frac{3}{p+1} = 2$

p. $\frac{1}{a-1} + \frac{3}{4} = \frac{4}{a-1}$ **q.** $\frac{2x}{x+1} + \frac{2}{3} = 2$ **r.** $\frac{x+1}{x+3} = \frac{4}{5}$

15.2 Simultaneous Equations

Consider the following pair of simultaneous equations

$$2x + y = 5$$

$$2x + 3y = 8$$

The coefficient of the unknown x in each of the equations is 2. That means, they are equal. We have seen how to solve simultaneous equations of this nature, i.e., when the coefficients of one unknown are equal. Let us see how simultaneous equations are solved when the coefficients of both unknowns are un equal, i.e., the coefficients of each unknown are different.

Example 1:

Sajithi and Sanjana have certain amounts of money. When twice the amount of money Sanjana has is added to the amount of money Sajithi has, we get Rs. 110. When thrice the amount of money Sanjana has is added to twice the amount of money Sajithi has, the amount is Rs. 190. Find the amount of money each has.

Let us see how simultaneous equations can be used to solve this problem. Let the amount of money Sajithi has be Rs. x and the amount of money Sanjana has be Rs. y. Then, the sum of the amount of money Sajithi has and twice the amount of money Sanjana has is Rs. $x + 2y$. Because this amount is equal to Rs. 110, we get

$$x + 2y = 110 \quad \text{——— } (1)$$

Also, the sum of twice the amount of money Sajithi has and thrice the amount of money Sanjana has is Rs. $2x + 3y$. Because this amount is equal to Rs 190, we get

$$2x + 3y = 190 \quad \text{——— } (2)$$

The coefficients of neither x nor of y of the equations (1) and (2) are equal. Therefore, let us equate the coefficients of one of the unknowns. In order to make the coefficient of x in the first equation 2, let us multiply that equation by 2. Then we get

$$\therefore \; 2x + 4y = 220 \quad \text{——— } (3)$$

Now, the coefficients of x in both equations are equal.
Therefore, from (3) and (2), we get,

$$2x + 4y - (2x + 3y) = 220 - 190$$
$$2x + 4y - 2x - 3y = 30$$
$$y = 30$$

Substituting the value of y in the first equation,

$$x + 2y = 110$$
$$x + 2 \times 30 = 110$$
$$x + 60 = 110$$
$$x = 110 - 60$$
$$x = 50$$

Therefore, the amount of money Sajithi has is Rs 50 and the amount of money Sanjana has is Rs 30.

Example 2

Solve: $2m + 3n = 13$
$3m + 5n = 21$

Let us denote the two equations as

$2m + 3n = 13$ ——— (1)
$3m + 5n = 21$ ——— (2)

From (1) $\times$ 3, we get, $6m + 9n = 39$ ——— (3)
From (2) $\times$ 2, we get, $6m + 10n = 42$ ——— (4)
Then, from (4) and (3), we get,

$$6m + 10n - (6m + 9n) = 42 - 39$$
$$6m + 10n - 6m - 9n = 3$$
$$n = 3$$

Substituting $n = 3$ in (1), we get

$$2m + 3n = 13$$
$$2m + 3\times 3 = 13$$
$$2m = 13 - 9$$
$$2m = 4$$
$$m = 2$$

i.e., $m = 2$ and $n = 3$

Example 3

The price of two oranges and one king coconut is Rs. 80. Three king coconuts cost the same amount as two oranges. Find the price of an orange and a king coconut separately.

Let us construct two equations from the given information.

Let the price of one orange be Rs. x and that of a king coconut be Rs. y.

Then, the price of two oranges and a king coconut is $2x + y$. Since this is equal to Rs. 80, we get

$$2x + y = 80.$$

Because the price of two oranges is equal to that of three king coconut, we get,

$$2x = 3y.$$

Let us denote the equations as

$$2x + y = 80 \quad \text{——— (1)}$$
$$2x = 3y \quad \text{——— (2)}$$

These two equations can be solved just the way we did in the previous example, by writing the second equation as $2x - 3y = 0$. However, there is an easier way to do it in this case. That is, substituting from one equation to the other.

Substitute $3y$ from equation (2) for $2x$ in equation (1) to get

$$3y + y = 80$$
$$4y = 80$$
$$y = 20$$

Substitute $y = 20$ in (1) to get

$$2x + 20 = 80$$
$$2x = 60$$
$$x = 30$$

Therefore, the price of one orange is Rs. 30 and that of a king coconut is Rs. 20.

Example 4

Solve: $x = 3y$

$$2x + 3y = 18$$

Denote the two equations as

$$x = 3y \quad \text{——— (1)}$$

$$2x + 3y = 18 \quad \text{——— (2)}$$

Substitute for x in equation (2) from equation (1) to get

$$2 \times (3y) + 3y = 18$$
$$6y + 3y = 18$$
$$9y = 18$$
$$y = 2$$

Substitute $y = 2$ in (1) to get

$$x = 3y$$
$$x = 3 \times 2$$
$$x = 6$$

i.e., $x = 6$ and $y = 2$.

Exercise 15.2

1. Solve each of the following pair of simultaneous equations.

(a) $x + 2y = 10$
$2x - 5y = 2$

(b) $x = 3y$
$x + 3y = 12$

(c) $2m + n = 5$
$m + 2n = 4$

(d) $3x + y = 14$
$2x + 3y = 21$

(e) $2x + 5y = 9$
$3x + 2y = 8$

(f) $4m - 3n = 7$
$7m - 2n = 22$

(g) $8x - 3y = 1$
$3x + 2y = 16$

(h) $6x + 5y = 5$
$9x - 4y = 19$

(*i*) $3x + 4y = 9$
$2x - 5y + 17 = 0$

(*j*) $3x - 4y = 8(2 - y) + 1$
$2(2x + 3y) = 26 - y$

2. The price of two baby shirts and three pairs of baby shorts is Rs. 1150. Three baby shirts and a pairs of baby shorts cost Rs. 850. Taking the price of a baby shirt as Rs. x and that of a pairs of baby shorts as Rs. y, construct two simultaneous equations and find the price of a baby shirt and that of a pairs of baby shorts separately by solving the two equations.

3. Dinithi's father tells Dinithi, "My age is now four times your age. 8 years earlier, I was twelve times older than you." Find the present ages of the father and Dinithi separately.

15.3 Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation where $a \neq 0$. The terms b and c might be 0. Observe the following equations.

(i) $x^2 + 5x + 6 = 0$
(ii) $2x^2 - 5x = 0$
(iii) $x^2 - 9 = 0$

Each of the above equations is a quadratic equation since $a \neq 0$, whereas $c = 0$ in the second equation and $b = 0$ in the third equation.

Before solving the quadratic equations, let us consider the following:

- When any number is multiplied by zero, the product is zero.
- If the product of two numbers is zero, then at least one of the numbers is zero.

So, let us see for what values of x the expression $(x-1)(x-3)$ is zero. The expression is zero only when $x - 1 = 0$ or $x - 3 = 0$, that is, only when $x = 1$ or $x = 3$.

Now, consider the equation $(x-1)(x-3) = 0$. The values $x = 1$ and $x = 3$ satisfy this

equation. Then, 1 and 3 are called the roots of this equation.
Now, consider the equation $x^2 + 5x + 6 = 0$.
Since $x^2 + 5x + 6 = (x + 3)(x + 2)$, the equation can be rewritten as $(x + 3)(x + 2) = 0$.
Then, $x + 3 = 0$ or $x + 2 = 0$.
There fore, $x = -3$ and $x = -2$ satisfy the equation. This can be verified as follows.
When $x = -3$,

$$\begin{aligned} x^2 + 5x + 6 &= (-3)^2 + 5(-3) + 6 \\ &= 9 + (-15) + 6 \\ &= 0 \end{aligned}$$

When $x = -2$,

$$\begin{aligned} x^2 + 5x + 6 &= (-2)^2 + 5(-2) + 6 \\ &= 4 + (-10) + 6 \\ &= 0 \end{aligned}$$

Thus, the roots of the equation $x^2 + 5x + 6 = 0$ are -3 and -2. In other words, the solutions of the equation are $x = -3$ and $x = -2$.

Example 1

Solve the equation $x^2 + 2x = 0$

$$\begin{aligned} x^2 + 2x &= 0 \\ x(x + 2) &= 0 \\ x = 0 \text{ or } x + 2 &= 0 \\ x = 0 \text{ or } x &= -2 \end{aligned}$$

Hence, $x = 0$ and $x = -2$ are the solutions of the given equation.

Example 2

Solve the equation $x^2 - 3x + 2 = 0$

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x - 1)(x - 2) &= 0 \\ x - 1 = 0 \text{ or } x - 2 &= 0 \\ x = 1 \text{ or } x &= 2 \end{aligned}$$

Hence, $x = 1$ and $x = 2$ are the solutions of the given equation.

Example 3

Solve the equation $x^2 - 4x - 21 = 0$

$$x^2 - 4x - 21 = 0$$

$$(x - 7)(x + 3) = 0$$

$$x - 7 = 0 \text{ or } x + 3 = 0$$

$$x = 7 \text{ or } x = -3$$

Hence, $x = 7$ and $x = -3$ are the solutions of the given equation.

Note: When there are two factors to a quadratic expression, there are two roots to the that equation.

Exercise 15.3

Solve each of the following quadratic equations.

(*a*) $(x - 2)(x - 3) = 0$

(*b*) $(x + 2)(x - 5) = 0$

(*c*) $(x - 4)(x - 4) = 0$

(*d*) $(x - 1)(2x - 1) = 0$

(*e*) $x(x + 3) = 0$

(*f*) $y(2y - 3) = 0$

(*g*) $x^2 - 16 = 0$

(*h*) $4x^2 - 1 = 0$

(*i*) $9x^2 - 27x = 0$

(*j*) $x^2 + 15x + 36 = 0$

(*k*) $2x^2 - 5x + 2 = 0$

(*l*) $2x^2 - 5x = 0$

(*m*) $2x^2 = 6x$

(*n*) $x^2 = 25$

(*o*) $(x + 3)^2 = 16$

(*p*) $x^2 = 9x + 36$

(*q*) $(2x - 3)^2 = 0$

(*r*) $2x^2 - 5x = 0$

(*s*) $(x - 1)(x - 2) = 2x^2 - 3x - 2$

(*t*) $\dfrac{x+3}{2} = \dfrac{3x+2}{x}$