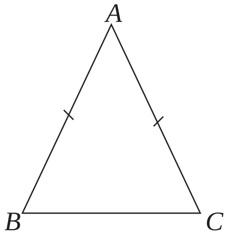
9 Triangles II

By studying this lesson you will be able to

solve problems by applying the theorem on isosceles triangles and its converse.

9.1 Isosceles triangles

If two sides of a triangle are equal, then it is called an isosceles triangle. The triangle ABC in the figure given below is an isosceles triangle. In this triangle, $AB = AC$. The angle in front of each side of a triangle is called the angle opposite that side.



The angle opposite the side AB is $A\hat{C}B$.
The angle opposite the side AC is $A\hat{B}C$.
The angle opposite the side BC is $B\hat{A}C$.
Further, A, which is the vertex at which the two equal sides meet is called the apex of the triangle.

A theorem related to isosceles triangles is given below.

Theorem: If two sides are equal in a triangle, the angles opposite the equal sides are equal.

According to the theorem, since $AB = AC$ in the above isosceles triangle ABC, $A\hat{B}C = A\hat{C}B$. Let us engage in the following activity to verify the truth of the above theorem.

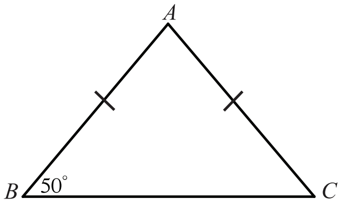
Activity

- Mark three points A, B and C (not collinear) such that $AB = AC = 5$cm.
- Complete the triangle ABC by joining the points A, B and C.
- Cut out the shape of the triangle ABC.
- Fold the triangular shaped piece of paper so that AB and AC coincide.
- Observe that $A\hat{B}C$ and $A\hat{C}B$ are equal.

Now let us consider several problems that can be solved by applying the above theorem together with theorems that have been learnt previously.

Example 1

In the triangle ABC, $AB = AC$ and $A\hat{B}C = 50^\circ$.
Find the magnitude of
(i) $A\hat{C}B$ (ii) $B\hat{A}C$



(i) $A\hat{C}B = A\hat{B}C$ ($AB = AC$, angles opposite equal sides in the isosceles triangle)
$\therefore A\hat{C}B = \underline{\underline{50^\circ}}$

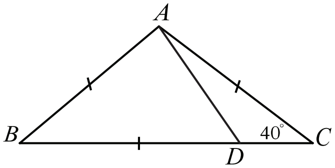
(ii) Since the sum of the interior angles of a triangle is 180°,

$$B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^\circ$$
$$\therefore B\hat{A}C + 50^\circ + 50^\circ = 180^\circ$$
$$\therefore B\hat{A}C = 180^\circ - (50^\circ + 50^\circ)$$
$$= 180^\circ - 100^\circ$$
$$= \underline{\underline{80^\circ}}$$

Example 2

In the triangle ABC, $AB = AC$ and $A\hat{C}B = 40^\circ$. The point D has been marked on the side BC such that $AB = BD$, and AD has been joined. Find separately the magnitude of each of the angles in the triangle ABD.

First, let us draw the figure with the given information.



According to the figure,
$A\hat{B}C = A\hat{C}B$ (Angles opposite equal sides in triangle ABC)
$\therefore A\hat{B}C = 40^\circ$
That is, $A\hat{B}D = 40^\circ$
In the triangle ABD,
$B\hat{A}D = B\hat{D}A$ ($AB = BD$)
$A\hat{B}D + B\hat{A}D + B\hat{D}A = 180^\circ$ (The sum of the interior angles of the triangle ABD is 180°)

$$40^\circ + 2B\hat{A}D = 180^\circ \text{ (Since } B\hat{A}D = B\hat{D}A\text{)}$$

$$2B\hat{A}D = 180^\circ - 40^\circ$$

$$2B\hat{A}D = 140^\circ$$

$$B\hat{A}D = 70^\circ$$

$$B\hat{D}A = 70^\circ \text{ (Since } B\hat{A}D = B\hat{D}A\text{)}$$

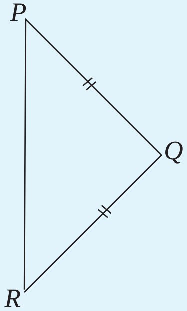
The magnitudes of the angles of the triangle *ABD* are 70º, 70º and 40º.

Do the following exercise by applying the theorem on isosceles triangles.

Exercise 9.1

1. Complete the following table by identifying all the isosceles triangles in the figures in each of the parts given below.

Figure	Triangle	Pair of equal sides	Pair of angles opposite equal sides
(i)	*PQR*	*PQ*, *RQ*	$Q\hat{P}R,\ Q\hat{R}P$
(ii)	*ACD*	*AD*, *DC*	$A\hat{C}D,\ D\hat{A}C$
	ABC		
(iii)	*KLM*		
	LMN		
(iv)	*PQR*		
	RST		
(v)	*ABD*		
	BCD		
	ABC		

(i)


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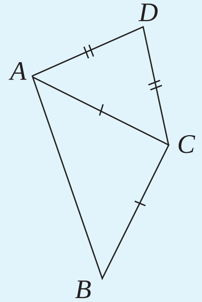

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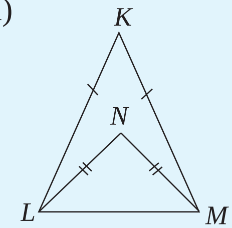

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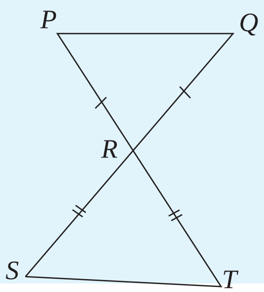

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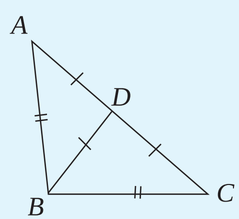
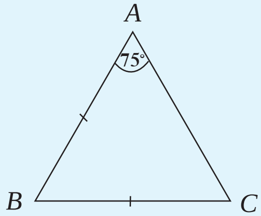


2. In each of the following triangles, the magnitude of one of the angles is given. Separately find the magnitude of the other two angles.

(i)



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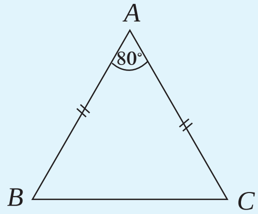

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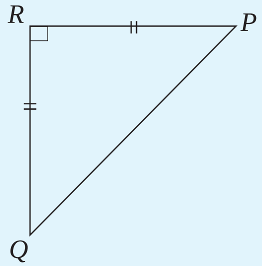
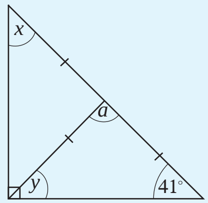


3. Find the value of each of the angles denoted by an unknown in the following figures.

(i)



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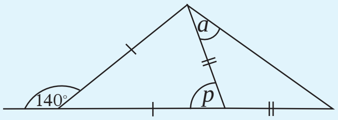

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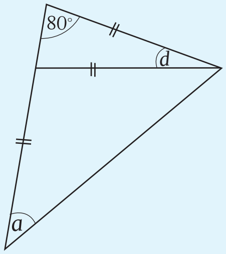

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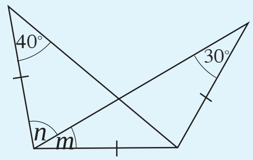

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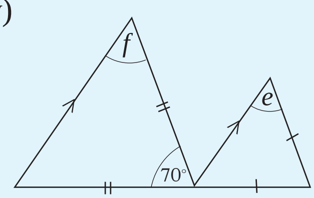
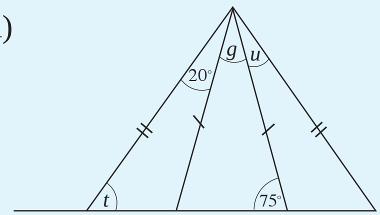
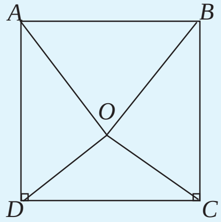


(vi)



4. In a certain isosceles triangle, the included angle of the two equal sides is 110^{o}. Find the magnitudes of the other two angles.

5. The point O lies within the square $ABCD$ such that AOB is an equilateral triangle. Find the magnitude of $D\hat{O}C$.



6. In the triangle ABE, while $\hat{A}$ is an obtuse angle, $AB = AE$. The point C lies on BE such that $AC = BC$. The bisector of the interior angle $C\hat{A}E$ meets BE at D.

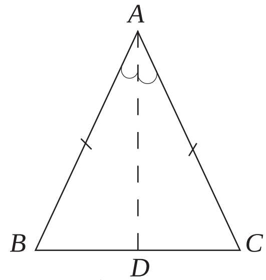
(i) Illustrate this information in a figure.

(ii) If $A\hat{B}C = 40^{\circ}$, find the magnitude of $D\hat{A}E$.

9.2 The formal proof of the theorem on isosceles triangles and applications of the theorem

Let us formally prove the theorem "In an isosceles triangle, the angles opposite equal sides are equal".

Data : $AB = AC$ in the triangle ABC
To prove that: $A\hat{B}C = A\hat{C}B$
Construction : Draw the bisector AD of the interior angle $B\hat{A}C$ such that it meets BC at D.



Proof: In the triangles ABD and ACD,

$AB = AC$ (Data)

$B\hat{A}D = D\hat{A}C$ (The bisector of $B\hat{A}C$ is AD)

AD is common to both the triangles

$\therefore \Delta ABD \equiv \Delta ACD$ (S A S)

Since the corresponding elements of congruent triangles are equal,

$A\hat{B}D = A\hat{C}D$

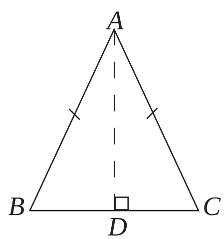
$\therefore A\hat{B}C = A\hat{C}B$

Now let us consider how several results on triangles are proved using the above theorem.

Example 1

$AB = AC$ in the triangle ABC in the figure.
Show that the following coincide.
(i) The perpendicular drawn from A to BC.
(ii) The bisector of the interior angle $B\hat{A}C$.
(iii) The straight line joining A and the midpoint of BC.
(iv) The perpendicular bisector of BC.

Let us first draw the perpendicular from the vertex A to the opposite side BC.
Construction : Draw the perpendicular from A to BC.
Proof : In the triangles ABD and ACD,



(i) $AB = AC$ (Data)

$A\hat{D}B = A\hat{D}C = 90^\circ$ (Construction)

AD is the common side

$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)

Since the corresponding elements of congruent triangles are equal,

(ii) $B\hat{A}D = C\hat{A}D$

i.e., AD is the bisector of $B\hat{A}C$.

(iii) $BD = DC$ (corresponding sides of congruent triangles)
Therefore AD is the line joining A and the mid point of BC.

(iv) $A\hat{D}B = A\hat{D}C = 90^\circ$ (construction)

Also $BD = DC$

Therefore, AD is the perpendicular bisector of BC.

In an isosceles triangle,

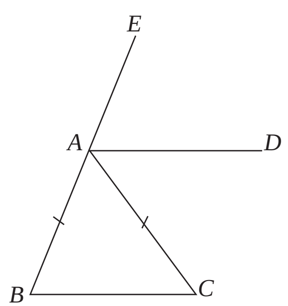
1. the perpendicular drawn from the apex to the opposite side
2. the bisector of the apex angle,
3. the straight line joining the apex to the midpoint of the opposite side and
4. the perpendicular bisector of the side opposite the apex,

coincide with each other.

In some instances, a result in geometry can be proved in several ways.
Now, let us consider such a result.

Example 2

$AB = AC$ in the triangle ABC. The side BA has been produced to E. The angle $C\hat{A}E$ is bisected by AD. Prove that AD and BC are parallel to each other.



To prove that $AD \parallel BC$, let us show that either a pair of alternate angles or a pair of corresponding angles are equal to each other.

Proof:

Method 1

In the triangle ABC, $A\hat{B}C = A\hat{C}B$ ($AB = AC$, angles opposite equal sides)

Since the side BA of the triangle ABC has been produced to E,

$$E\hat{A}C = A\hat{B}C + A\hat{C}B \quad \text{(Theorem on the exterior angle)}$$

$$E\hat{A}C = 2A\hat{C}B \quad (\text{Since } A\hat{B}C = A\hat{C}B) \text{ ——— } (1)$$

From the figure, $E\hat{A}C = E\hat{A}D + D\hat{A}C$

$$E\hat{A}C = 2D\hat{A}C \quad (\text{Since } AD \text{ is the bisector of } E\hat{A}C) \text{ ——— } (2)$$

$$2A\hat{C}B = 2D\hat{A}C$$

$$\therefore \quad A\hat{C}B = D\hat{A}C$$

This is a pair of alternate angles. Since a pair of alternate angles are equal, AD is parallel to BC.

Method 2

According to the above figure, $A\hat{B}C$ and $E\hat{A}D$ are a pair of corresponding angles. By showing that these two angles are equal in the same manner as above, we can show that $BC \parallel AD$.

Method 3

The above proof could also be given as follows using algebraic symbols.

In the triangle ABC, let $A\hat{B}C = x$ ——— (1)

$$A\hat{B}C = A\hat{C}B \quad (\text{Since } AB = AC)$$

$$\therefore \quad A\hat{C}B = x$$

Since the side BA of the triangle ABC has been produced to E,

$$E\hat{A}C = A\hat{B}C + A\hat{C}B \quad \text{(Theorem on the exterior angle)}$$

$$E\hat{A}C = x + x$$

$$= 2x$$

$$E\hat{A}D = x \quad (\text{Since } AD \text{ is the bisector of } E\hat{A}C) \text{ ——— } (2)$$

From (1) and (2), $E\hat{A}D = A\hat{B}C$

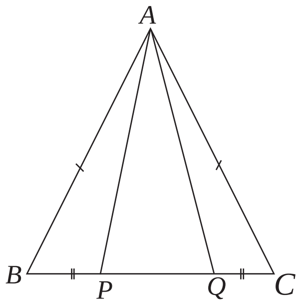
This is a pair of corresponding angles. Since a pair of corresponding angles are equal, $AD \parallel BC$.

Example 3

In the triangle ABC, $AB = AC$. The points P and Q lie on the side BC such that $BP = CQ$. Prove that

(i) $\Delta\, APB \equiv \Delta\, AQC$

(ii) $A\hat{P}Q = A\hat{Q}P$



Proof : In the triangles APB and ACQ,

$AB = AC$ (Data)

$A\hat{B}P = A\hat{C}Q$ (Since $AB = AC$)

$BP = CQ$ (Data)

$\therefore \Delta\, ABP \equiv \Delta\, AQC$ (SAS)

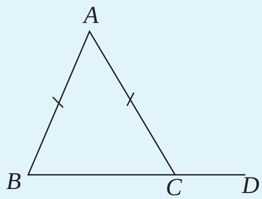
(ii) Since $\Delta\, ABP \equiv \Delta\, AQC$, $AP = AQ$ (Corresponding elements of congruent triangles)

In the triangle APQ,

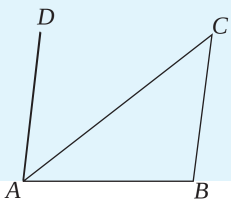
$A\hat{P}Q = A\hat{Q}P$ ($AP = AQ$, angles opposite equal sides)

Do the following exercise by applying the above theorem on isosceles triangles, and the other theorems learnt previously.

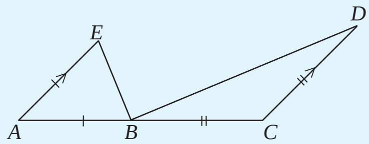
Exercise 9.2

1. 

Based on the information in the figure, prove that $A\hat{B}C + A\hat{C}D = 180^\circ$

2. 

In the figure, $AB = BC$ and $AD//BC$. Prove that the bisector of $D\hat{A}B$ is AC.

3. 

In the given figure, ABC is a straight line. Provide answers based on the information in the figure.

(i) Find the value of $B\hat{A}E + B\hat{C}D$

(ii) Show that $D\hat{B}E = 90^\circ$

4. D is the midpoint of the side BC of the triangle ABC. If $BD = DA$, prove that $B\hat{A}C$ is a right angle.

5. $AB = AC$ in the triangle ABC. The points P, Q and R lie on the sides AB, BC and AC respectively, such that $BP = CQ$ and $BQ = CR$.

(i) Draw a figure with this information included in it.

(ii) Prove that $\Delta PBQ \equiv \Delta QRC$.

(iii) Prove that $Q\hat{P}R = Q\hat{R}P$.

6. In the triangle ABC, $\hat{B}$ is a right angle. BD has been drawn perpendicular to AC. The point E lies on AC such that $CE = CB$.

(i) Draw a figure with this information included in it.

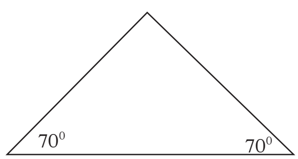
(ii) Prove that $A\hat{B}D$ is bisected by BE.

7. Prove that the angles in an equilateral triangle are 60°.

9.3 The converse of the theorem on isosceles triangles

Let us now examine whether the sides opposite equal angles are equal for a triangle.

Activity



- Draw a line segment of length 5 cm and mark an angle of 70° at one end point using a protractor.
- Draw an angle of 70° at the other endpoint of the line segment too.
- Produce the two sides of the angles that were drawn, until they meet.
- Then a figure such as the above triangle is obtained.
- Cut out the triangle and fold it such that the equal angles coincide.
- Now identify the sides of the triangle which are equal.
- What is the special property that can be mentioned about the sides opposite the equal angles of the triangle?
- Draw several more triangles in the above manner, each time changing the

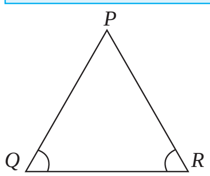
magnitude of the equal angles, cut them out and see whether the above property is true for these triangles too.

- Observe that the sides opposite equal angles of a triangle are equal to each other.

The result that was obtained through the above activity is true in general and is given as a theorem below.

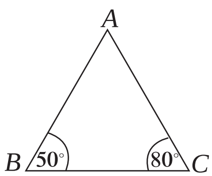
Theorem (Converse of the theorem on isosceles triangles):

The sides opposite equal angles of a triangle are equal.



According to the theorem, if $P\hat{Q}R = P\hat{R}Q$ in the triangle PQR, then $PR = PQ$.

Example 1



Find the equal sides of the triangle ABC in the figure.

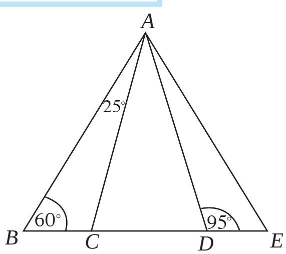
In the triangle ABC,

$$\hat{A}+\hat{B}+\hat{C} = 180^\circ \text{ (Sum of the interior angles of the triangle)}$$
$$\hat{A} + 50^\circ + 80^\circ = 180^\circ$$
$$\hat{A} = 180^\circ - (50^\circ + 80^\circ)$$
$$= 180^\circ - 130^\circ$$
$$= 50^\circ$$

$\hat{A} = \hat{B}$

$\therefore \underline{\underline{AC = BC}}$ (Sides opposite equal angles of a triangle)

Example 2



Based on the information in the figure, show that $AC = AD$.

Consider the triangle ABC.

$$A\hat{C}D = A\hat{B}C + B\hat{A}C \quad \text{(Exterior angle = sum of the interior opposite angles)}$$
$$= 60^\circ + 25^\circ$$
$$= 85^\circ$$

Since CDE is a straight line,

$A\hat{D}C + A\hat{D}E = 180^\circ$ (Adjacent angles on a straight line)

$A\hat{D}C = 180^\circ - 95^\circ$

$= 85^\circ$

In the triangle ACD,

$A\hat{C}D = 85^\circ$

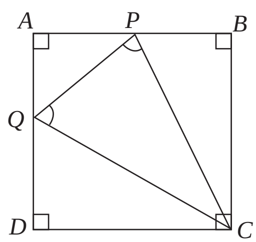
$A\hat{D}C = 85^\circ$

$\therefore \quad A\hat{D}C = A\hat{C}D$

$\therefore \quad AC = AD$ (Sides opposite equal angles)

Example 3

In the square $ABCD$, the points P and Q lie on the sides AB and AD respectively, such that $Q\hat{P}C = P\hat{Q}C$. Prove that $BP = QD$.



In the triangle PQC,

$Q\hat{P}C = P\hat{Q}C$ (Data)

$\therefore QC = PC$ (Sides opposite equal angles)

In the triangles PBC and DQC,

$P\hat{B}C = Q\hat{D}C = 90^\circ$ (Vertex angles of the square)

$BC = DC$ (Sides of the square)

$CP = CQ$ (Proved)

$\therefore \Delta PBC \equiv \Delta DQC$ (RHS)

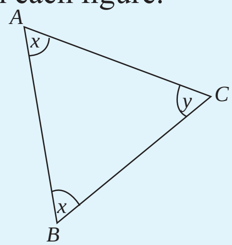
Since the corresponding elements of congruent triangles are equal,

$BP = QD$

Exercise 9.3

1. Select the isosceles triangles from the following figures, based on the information in each figure.

(i)



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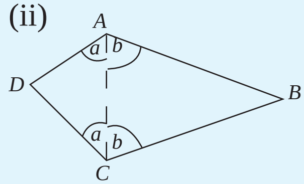

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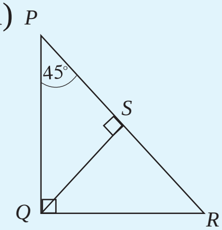

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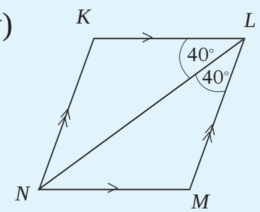
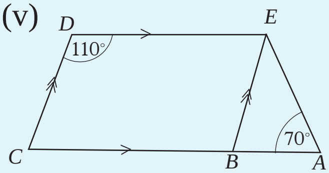
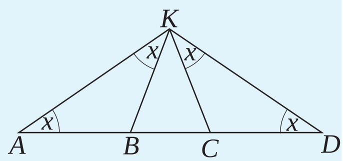


(v)

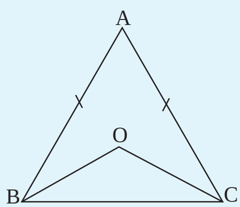


(vi)



2. If $A\hat{B}C = B\hat{C}A = B\hat{A}C$ in the triangle ABC, prove that it is an equilateral triangle.

3.

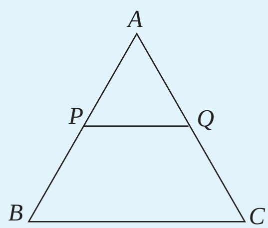


$AB = AC$ in the given triangle. The bisectors of the angles $A\hat{B}C$ and $A\hat{C}B$ meet at O. Prove that BOC is an isosceles triangle.

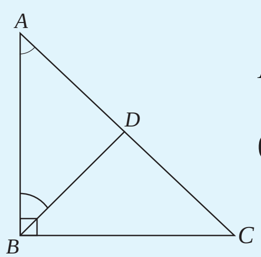
4. In the figure, $AB = AC$ and $BC//PQ$. Prove that

(i) $AP = AQ$

(ii) $BP = CQ$



5.



The point D lies on the side AC of the figure, such that $B\hat{A}D = D\hat{B}A$. Also $A\hat{B}C = 90^{\circ}$. Prove that,

(i) $D\hat{B}C = D\hat{C}B$

(ii) D is the midpoint of AC.

6. The bisectors of the angles $\hat{B}$ and $\hat{C}$ of the triangle ABC, meet at the point R. The points P and Q lie on AB and AC respectively such that PQ passes through R and is parallel to BC. Prove that,

(i) $PB = PR$

(ii) $PQ = PB + QC$.

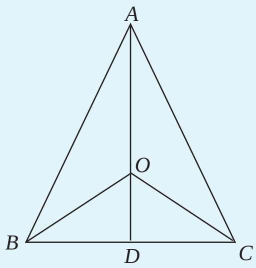
7. In the triangle ABC, the point P lies on AC such that $A\hat{C}B = A\hat{B}P$. The bisector of $P\hat{B}C$ meets the side AC at Q. Prove that $AB = AQ$.

8. In the quadrilateral $PQRS$, $PQ = SR$. The diagonals PR and QS which are equal in length intersect at T. Prove that,

(i) $\Delta PQR \equiv \Delta SQR$

(ii) $QT = RT$.

9.



In the triangle ABC, $AB = AC$. The bisectors of $A\hat{B}C$ and $A\hat{C}B$ meet at O. AO produced meets BC at D. Prove that,

(i) BOC is an isosceles triangle.

(ii) $\Delta AOB \equiv \Delta AOC$

(iii) AD is perpendicular to BC.

10. A circle of centre O is given in the figure. Show that $B\hat{O}C = 2B\hat{A}C$