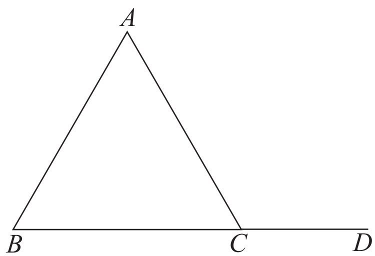
8 Triangles I

By studying this lesson you will be able to

- prove riders using the theorems related to the angles of a triangle.

8.1 The interior and exterior angles of a triangle

The angles $B\hat{A}C$, $A\hat{B}C$ and $A\hat{C}B$ which are within the triangle ABC are called the interior angles of the triangle ABC (in short, these are called the angles of the triangle ABC).



The side BC of the triangle ABC has been produced up to D as indicated in the figure. The angle $A\hat{C}D$ which is then formed is an **exterior angle** of the triangle. Since BCD is a straight line, $A\hat{C}B$ is an adjacent supplementary angle of $A\hat{C}D$. The angles $B\hat{A}C$ and $A\hat{B}C$ which are the interior angles of triangle ABC apart from $A\hat{C}B$ are called the **interior opposite angles** of the exterior angle $A\hat{C}D$. In the same manner, there are pairs of interior opposite angles corresponding to the exterior angles formed by producing the other two sides of the triangle too.

The following theorem specifies a relationship that exists between an exterior angle of a triangle and its interior opposite angles.

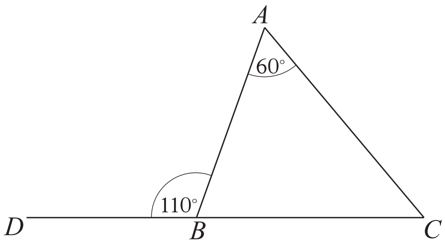
Theorem: The exterior angle formed when a side of a triangle is produced is equal to the sum of the two interior opposite angles.

Accordingly, for the above triangle ABC,

$$A\hat{C}D = A\hat{B}C + B\hat{A}C$$

Let us now consider how problems are solved using this theorem.

Example 1



Find the magnitude of $A\hat{C}B$ based on the information in the figure.

According to the theorem,

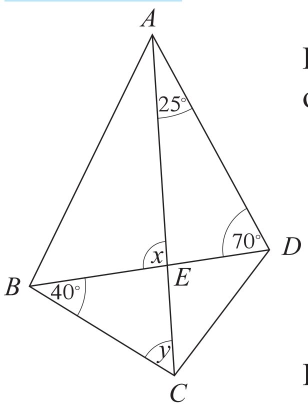
$B\hat{A}C + A\hat{C}B = A\hat{B}D$ (The exterior angle is equal to the sum of the interior opposite angles)

$\therefore\ 60^\circ + A\hat{C}B = 110^\circ$

$\therefore\quad A\hat{C}B = 110^\circ - 60^\circ$

$\underline{\underline{A\hat{C}B = 50^\circ}}$

Example 2



Based on the information in the figure, find the magnitudes of $A\hat{E}B$ and $B\hat{C}E$.

Let us denote $A\hat{E}B$ by x and $B\hat{C}E$ by y.

It is clear that $A\hat{E}B$ is an exterior angle of the triangle AED

Accordingly, $x = 25^\circ + 70^\circ$ (The exterior angle is equal to the sum of the interior opposite angles)

$= 95^\circ$

Since $A\hat{E}B$ is an exterior angle of the triangle BCE

$y + 40^\circ = x$ (The exterior angle is equal to the sum of the interior opposite angles)

$\therefore\ y + 40^\circ = 95^\circ$

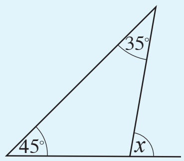
$\therefore\ y = 95^\circ - 40^\circ$

$\underline{\underline{y = 55^\circ}}$

Exercise 8.1

1. Find the magnitudes of the angles represented by the letters in each of the figures.

(i)



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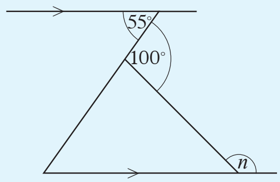

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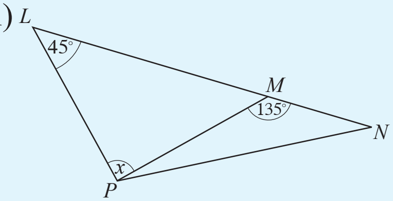

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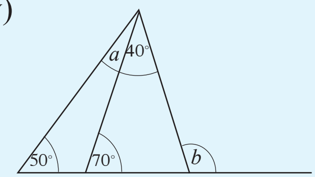
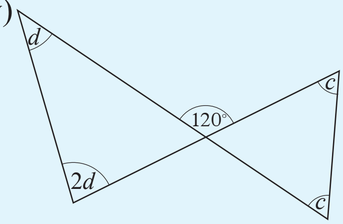


(v)



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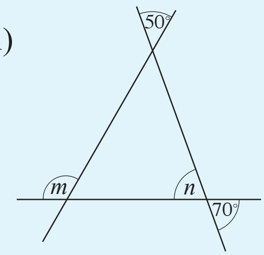
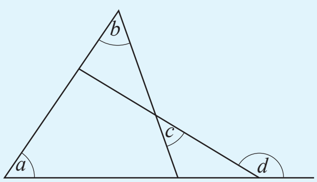
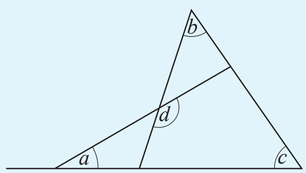


2. Express d in terms of a, b and c based on the information in each figure.

(i)

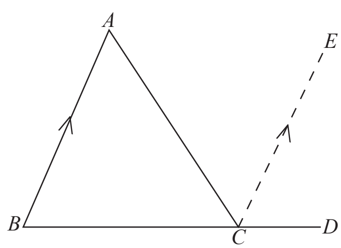


(ii)



8.2 The formal proof and applications of the theorem on the exterior angle of a triangle

Formal Proof:



Data : The side BC of the triangle ABC has been produced up to D.

To be proved: $A\hat{C}D = A\hat{B}C + B\hat{A}C$

Construction: CE is drawn parallel to BA.

Proof:

$$E\hat{C}D = A\hat{B}C \quad (BA \,//\, CE \text{ since corresponding angles})$$

$$A\hat{C}E = B\hat{A}C \quad (BA \,//\, CE \text{ since alternate angles})$$

From ① and ②

$$E\hat{C}D + A\hat{C}E = A\hat{B}C + B\hat{A}C \text{ (Axiom)}$$

However according to the figure, the sum of the adjacent angles $E\hat{C}D$ and $A\hat{C}E$ is $A\hat{C}D$.

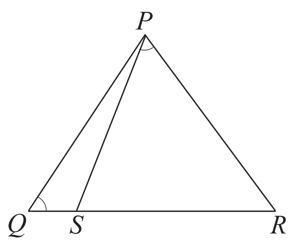
$$\therefore \underline{\underline{A\hat{C}D = A\hat{B}C + B\hat{A}C}}$$

Let us now prove several riders by using the formally proved exterior angle theorem and the other theorems you have learnt so far.

Example 1

The point S is located on the side QR of the triangle PQR such that $P\hat{Q}S = S\hat{P}R$. Prove that $Q\hat{P}R = P\hat{S}R$.

Let us first draw a sketch and include the given information in it.



Proof:

$P\hat{S}R$ is an exterior angle of the triangle PQS.

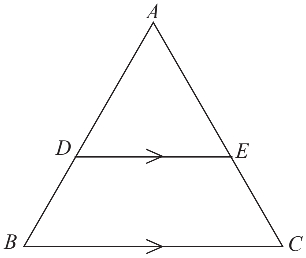
$$\therefore \; Q\hat{P}S + P\hat{Q}S = P\hat{S}R$$

$$\therefore \; Q\hat{P}S + S\hat{P}R = P\hat{S}R \quad (\text{Since } P\hat{Q}S = S\hat{P}R)$$

$$\text{But, } Q\hat{P}S + S\hat{P}R = Q\hat{P}R \quad (\text{Adjacent angles})$$

$$\therefore \; \underline{\underline{Q\hat{P}R = P\hat{S}R}} \quad (\text{Axiom})$$

Example 2



Based on the information in the figure, prove that $B\hat{A}C + A\hat{B}C = D\hat{E}C$

$C\hat{E}D$ is an exterior angle of the triangle ADE.

$\therefore \; D\hat{E}C = D\hat{A}E + A\hat{D}E$

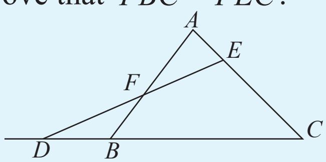
Also $D\hat{A}E$ and $B\hat{A}C$ are the same angle.

$\therefore \; A\hat{D}E = A\hat{B}C$ (Corresponding angles, $DE \,//\, BC$)

$\therefore \; \underline{\underline{D\hat{E}C = B\hat{A}C + A\hat{B}C}}$

Exercise 8.2

1. If $B\hat{D}F = E\hat{A}F$ in the figure given below, fill in the blanks in the following statements to prove that $F\hat{B}C = F\hat{E}C$.



Proof: Since $F\hat{B}C$ is an exterior angle of the triangle DBF,

$F\hat{B}C =$ ...... + ...

But $B\hat{F}D =$ ..... (Vertically opposite angles)

and $B\hat{D}F =$ ..... (................)

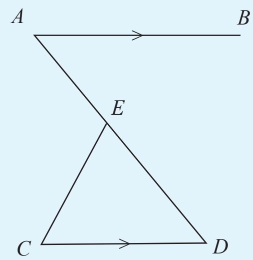
$\therefore \; F\hat{B}C =$ ..... + ...

But $F\hat{E}C$ is an exterior angle of the triangle AFE.

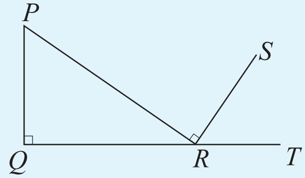
$\therefore \; F\hat{E}C =$ ..... + ... (................)

$\therefore \; \underline{\underline{F\hat{B}C = F\hat{E}C}}$

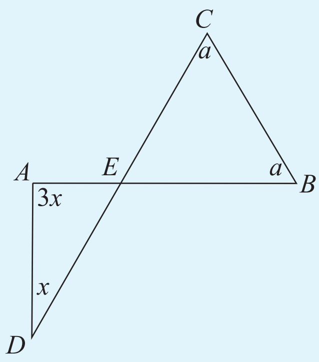
2. As indicated in the figure, AB and CD are parallel to each other. Prove that $A\hat{E}C = B\hat{A}D + E\hat{C}D$.



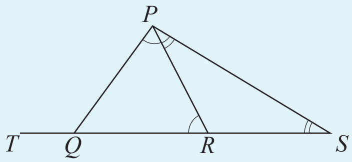
3. As indicated in the figure, $P\hat{Q}R$ and $P\hat{R}S$ are right angles. If QRT is a straight line, prove that $Q\hat{P}R = S\hat{R}T$.



4. The lines AB and CD intersect each other at point E as indicated in the figure. Based on the information in the figure, show that $a = 2x$.



5. In the given figure, $P\hat{R}Q = Q\hat{P}R$ and $R\hat{P}S = P\hat{S}R$. Based on the information in the figure, show that $P\hat{Q}T = 4\,P\hat{S}R$. (Hint: Take $P\hat{S}R = x$)



6. PQR is a triangle. RS is perpendicular to PQ and QT is perpendicular to PR. (S and T are on PQ and PR respectively). The lines SR and QT intersect at U. Prove that $S\hat{Q}U = T\hat{R}U$.

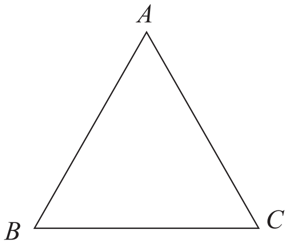
7. The side BC of the triangle ABC has been produced to E. The straight line AD has been drawn such that it meets the side CE at the point D and such that $B\hat{A}C = C\hat{A}D$. Also, $A\hat{B}C = B\hat{A}C$.

Prove that,

(i) $A\hat{C}D = 2\,A\hat{B}C$ and

(ii) $A\hat{D}E = 3\,A\hat{B}C$.

8.3 Theorem related to the interior angles of a triangle



The interior angles of the triangle ABC are $A\hat{B}C$, $B\hat{A}C$ and $A\hat{C}B$. We know that the sum of the magnitudes of these angles is equal to 180°.

This is expressed as a theorem as follows.

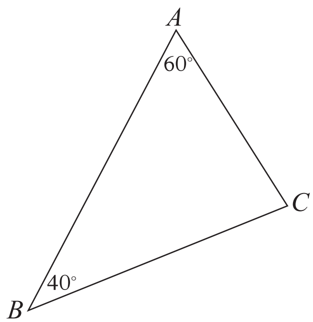
Theorem: The sum of the interior angles of a triangle is 180°

Accordingly, in relation to the above triangle, $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ$

Let us now consider how problems are solved using this theorem.

Example 1

Find the magnitude of $A\hat{C}B$ based on the information in the figure.



$B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^\circ$ (The sum of the interior angles)

$\therefore\ 60^\circ + 40^\circ + A\hat{C}B = 180^\circ$

$\therefore\ \underline{\underline{A\hat{C}B = 80^\circ}}$

Example 2

Find the value of x based on the information in the figure.

Since the sum of the interior angles of the triangle is 180°,

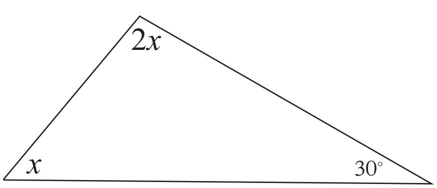
$\therefore x + 2x + 30^\circ = 180^\circ$

$\therefore\ 3x + 30^\circ = 180^\circ$

$\therefore\ 3x = 180^\circ - 30^\circ$

$3x = 150^\circ$

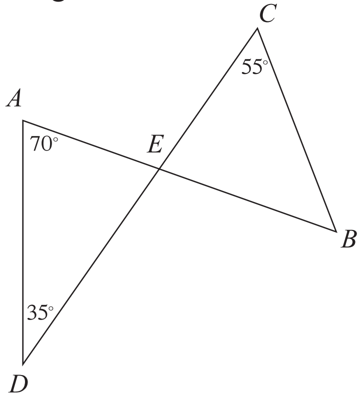
$\therefore\ \underline{\underline{x = 50^\circ}}$



Example 3

The straight lines AB and CD intersect at E. If $A\hat{D}E = 35^\circ$, $D\hat{A}E = 70^\circ$ and $E\hat{C}B = 55^\circ$ find the magnitude of $C\hat{B}E$.

First draw a figure with the given information.



According to the given information, considering the triangle ADE,

$A\hat{D}E + D\hat{A}E + A\hat{E}D = 180^\circ$ (Sum of the interior angles of a triangle)

$35^\circ + 70^\circ + A\hat{E}D = 180^\circ$

$\therefore \quad A\hat{E}D = 180^\circ - 105^\circ$

$= 75^\circ$

But, $A\hat{E}D = B\hat{E}C$ (Vertically opposite angles)

$B\hat{E}C = 75^\circ$

In the triangle BEC,

$B\hat{E}C + B\hat{C}E + C\hat{B}E = 180^\circ$ (Sum of the interior angles of a triangle)

$C\hat{B}E = 180^\circ - (75^\circ + 55^\circ)$

$= 180^\circ - 130^\circ$

$= \underline{\underline{50^\circ}}$

Example 4

Determine whether there can be a triangle with interior angles 55°, 60° and 75°.

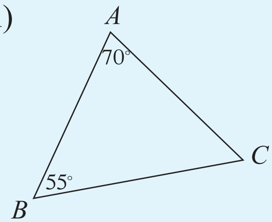
$55^\circ + 60^\circ + 75^\circ = 190^\circ$.

The sum of the interior angles of any triangle is 180°. Since the sum of the above three angles is not equal to 180°, there cannot be a triangle with the given angles as the interior angles.

Exercise 8.3

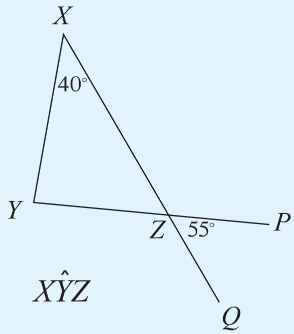
1. Using the information in each figure, determine the magnitude of the angle given below the figure.

(i)



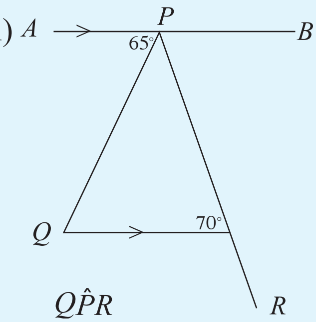
$A\hat{C}B$

(ii)



$X\hat{Y}Z$

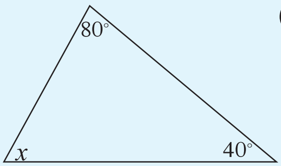
(iii)



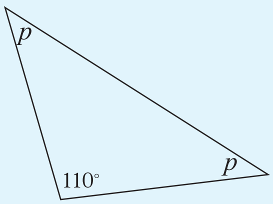
$Q\hat{P}R$

2. Using the given information, find the magnitude of each of the angles denoted by a letter in the following figures.

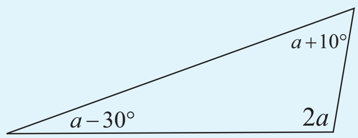
(i)



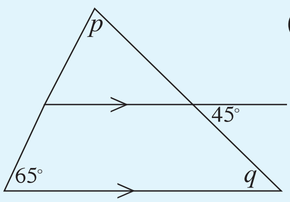
(ii)



(iii)



(iv)



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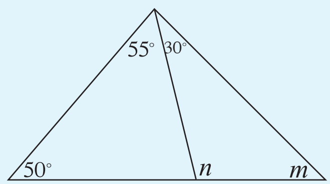

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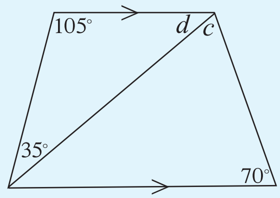


3. For each of the triplets of angles given below, determine whether they can be the interior angles of a triangle.

(i) $50^\circ, 40^\circ, 90^\circ$ (ii) $70^\circ, 30^\circ, 75^\circ$ (iii) $55^\circ, 72^\circ, 58^\circ$

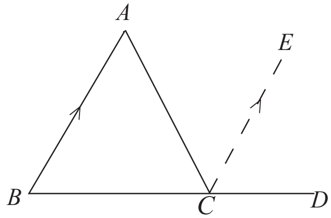
(iv) $60^\circ, 60^\circ, 60^\circ$ (v) $100^\circ, 20^\circ, 65^\circ$ (vi) $53^\circ, 49^\circ, 78^\circ$

4. The interior angles of a triangle are in the ratio 2 : 3 : 4. Find the magnitude of each of the angles.

5. The magnitude of the largest angle of a triangle is three times the magnitude of the smallest angle and the magnitude of the other angle is twice as large as the smallest angle. Find the magnitude of each of the angles.

8.4 Formal proof and applications of the theorem that the sum of the interior angles of a triangle is 180°.

The formal proof of the theorem "The sum of the interior angles of a triangle is 180°" is given below.



Data : ABC is a triangle

To be proved : $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ$

Construction : Produce BC up to D and draw CE parallel to BA

Proof: $A\hat{B}C = E\hat{C}D$ (Corresponding angles, $BA // CE$) --------------- (1)

$B\hat{A}C = A\hat{C}E$ (Alternate angles, $BA // CE$) --------------- (2)

From (1) and (2),

$$A\hat{B}C + B\hat{A}C = E\hat{C}D + A\hat{C}E$$

By adding $A\hat{C}B$ to both sides,

$$A\hat{B}C + B\hat{A}C + A\hat{C}B = E\hat{C}D + A\hat{C}E + A\hat{C}B$$

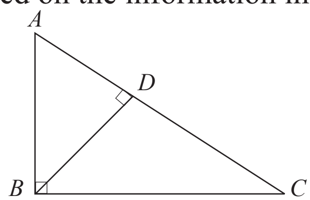
But,

$$E\hat{C}D + A\hat{C}E + A\hat{C}B = 180^\circ \text{ (Angles on the straight line } BCD\text{)}$$

$$\underline{\underline{A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ}}$$

Example 1

Prove that $A\hat{B}D = B\hat{C}D$ based on the information in the figure.



In the triangle BDC,

$$B\hat{D}C = 90^\circ \text{ (Given)}$$

Also $B\hat{D}C + D\hat{B}C + B\hat{C}D = 180^\circ$ (The sum of the interior angles of a triangle)

$$90^\circ + D\hat{B}C + B\hat{C}D = 180^\circ$$

$$D\hat{B}C + B\hat{C}D = 180^\circ - 90^\circ$$

$$= 90^\circ \text{ ——— ①}$$

In the triangle ABC,

$$A\hat{B}C = 90^\circ \text{ (Given)}$$

Since $A\hat{B}C = A\hat{B}D + D\hat{B}C$,

$$A\hat{B}D + D\hat{B}C = 90^\circ \text{ ——— ②}$$

By (1) and (2)

$$D\hat{B}C + B\hat{C}D = A\hat{B}D + D\hat{B}C$$

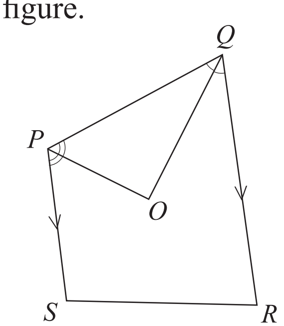
By subtracting $D\hat{B}C$ from both sides

$$\underline{\underline{B\hat{C}D = A\hat{B}D}}$$

Example 2

In the quadrilateral $PQRS$, the sides PS and QR are parallel to each other. The bisectors of the interior angles P and Q meet at O. Prove that $P\hat{O}Q$ is a right angle.

First let us draw the relevant figure.



Proof: Since $PS//QR$

$$S\hat{P}Q + P\hat{Q}R = 180^\circ \text{ (Allied angles)}$$

By dividing both sides by two

$$\frac{1}{2}SPQ + \frac{1}{2}PQR = \frac{180^\circ}{2} \text{ (Axiom)}$$

Since PO is the bisector of $S\hat{P}Q$ and QO is the bisector of $P\hat{Q}R$,

$$\frac{1}{2}S\hat{P}Q = Q\hat{P}O \text{ and}$$

$$\frac{1}{2}P\hat{Q}R = P\hat{Q}O$$

$$\therefore \quad Q\hat{P}O + P\hat{Q}O = 90^\circ$$

In the triangle POQ,

$$P\hat{O}Q + Q\hat{P}O + P\hat{Q}O = 180^\circ \text{ (Sum of the interior angles of a triangle)}$$

$$P\hat{O}Q + 90^\circ = 180^\circ$$

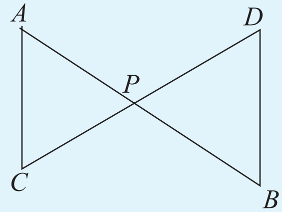
$$P\hat{O}Q = 90^\circ$$

$\therefore$ $\underline{\underline{P\hat{O}Q \text{ is a right angle.}}}$

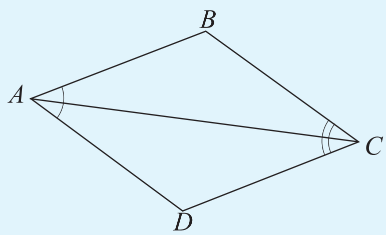
Now engage in the following exercise which contains problems involving proofs.

Exercise 8.4

1. In the given figure, $A\hat{C}P = P\hat{B}D$. Prove that $C\hat{A}P = P\hat{D}B$.



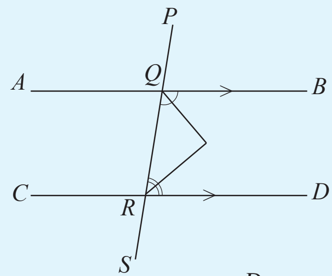
2. In the given figure, the diagonal AC of the quadrilateral $ABCD$ bisects the angles $B\hat{A}D$ and $B\hat{C}D$. Prove that $A\hat{B}C = A\hat{D}C$.



3. In the given figure, AB and CD are parallel lines. The bisectors of the angles $B\hat{Q}R$ and $Q\hat{R}D$ meet at O.

(i) Find the value of $O\hat{Q}R + Q\hat{R}O$.

(ii) Prove that $Q\hat{O}R$ is a right angled triangle.

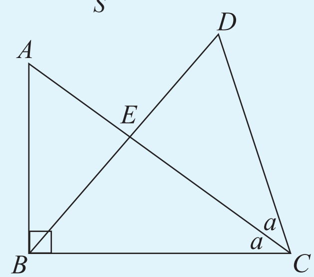


4. Using the information in the figure

(i) write down the magnitude of $B\hat{A}E$, in terms of a.

(ii) write down the magnitude of $B\hat{D}C + D\hat{B}C$ in terms of a.

(iii) Show that $B\hat{D}C + D\hat{B}C = 2\,B\hat{A}E$.



5 In the triangle ABC, $\hat{A} = \hat{B} = \hat{C}$. The bisector of $B\hat{A}C$ meets the side CB at D.

(i) Find the magnitude of $B\hat{A}C$.

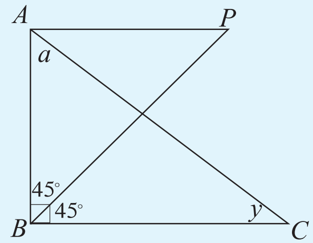
(ii) Prove that ABD is a right angled triangle.

Miscellaneous Exercise

1. If $\hat{A} + \hat{B} = 110^\circ$, and $\hat{B} + \hat{C} = 120^\circ$ in the triangle ABC, find the magnitude of each angle of the triangle.

2. The magnitude of a $B\hat{A}C$ of the triangle ABC is 100°. The bisectors of the interior angles $A\hat{B}C$ and $A\hat{C}B$ meet at O. Find the magnitude of $B\hat{O}C$.

3. In the figure, the straight line drawn from the point A perpendicular to the side BA of the triangle ABC meets the bisector of $A\hat{B}C$ at P. Prove that $B\hat{A}C + A\hat{C}B = 2\,A\hat{P}B$.



4. $A\hat{C}B = 3\,A\hat{B}C$ in the triangle ABC. The bisector of $B\hat{A}C$ meets BC at E. The point D lies on AE produced such that $AD \perp BD$. Prove that BC is the bisector of $A\hat{B}D$ (Hint: Take $A\hat{B}C = x$ and $B\hat{A}C = 2a$).

5. The line PQ has been drawn through the point A, parallel to the side BC of the triangle ABC. Prove that the sum of the interior angles of the triangle ABC is 180°.