7 Factors of Quadratic Expressions

By studying this lesson you will be able to

- find the factors of trinomial quadratic expressions,
- find the factors of the difference of two squares.

Factors of algebraic expressions

We know that $2x + 6$ is a binomial algebraic expression. Since it can be expressed as $2(x + 3)$, we also know that 2 and $x + 3$ are its factors.

Similarly, since $4x^2 + 6x = 2x\,(2x + 3)$, we know that 2, x and $2x + 3$ are the factors of $4x^2 + 6x$.
Now let us find the factors of $a^2 - 2a + ab - 2b$

$$\begin{aligned} a^2 - 2a + ab - 2b &= a\,(a - 2) + b\,(a - 2) \\ &= (a - 2)(a + b) \end{aligned}$$

Therefore, the factors of $a^2 - 2a + ab - 2b$ are $a - 2$ and $a + b$.

Do the following exercise to further recall what has been learnt earlier about factoring algebraic expressions.

Review Exercise

1. Write each of the following algebraic expressions as a product of its factors.

A. **a.** $3x + 12$ **b.** $p^2 - p$ **c.** $x^2 + 3xy$
d. $2a - 4a^2$ **e.** $p^2q - pq$ **f.** $2pq - 4p^2q$
g. $3m^2n + n^2$ **h.** $2a^2 - 4ab$ **i.** $2a^2 - 8ab - 2b^2$
j. $5x^2 - 10x^2y^2 - 15x^2\ y$ **k.** $3x^2y - 6x^2y^2 + 6xy^2$ **l.** $a^2bc + ab^2c - abc^2$

B. **a.** $x(a + b) + y(a + b)$ **b.** $2a(3x + y) - b(3x + y)$
c. $p(2a - 3b) + q(2a - 3b)$ **d.** $2(x - 3) - xy + 3y$
e. $3b + 3 + a(b + 1)$ **f.** $x^2 - xy + 4x - 4y$
g. $a^2 - 2ab - 5a + 10b$ **h.** $m - 3mn - n + 3n^2$

2. Fill in the blanks in (i) and (ii), and factor the expressions given below them accordingly.

(i) $a(2x - y) + b(y - 2x)$
$= a(2x - y) - b(........)$
$= \underline{\underline{(........)(........)}}$

(ii) $p(a - b) - q(b - a)$
$= p(a - b) .... q(a - b)$
$= \underline{\underline{(a - b)(........)}}$

a. $x(2p - q) - y(q - 2p)$
b. $3x(2a - b) + 2y(b - 2a)$
c. $m(l - 2n) - p(2n - l)$
d. $k(2x + y) - l(y + 2x)$
e. $a(x + 3y) - b(-x - 3y)$
f. $b(m - 2n) + d(2n - m)$

Defining trinomial quadratic expressions

Now let us consider factoring quadratic expressions such as $x^2 + 2x - 3$. This expression is of the form $ax^2 + bx + c$. Here a, b and c take integer values. We call an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are non-zero, a trinomial quadratic expression in x. a is said to be the coefficient of x^2, b the coefficient of x and c the constant term. It is easy to find the factors of such an expression when the terms are written in this order. The coefficient of x^2 in $x^2 + 2x - 3$ is 1 while the coefficient of x is 2 and the constant term is -3. $4 + 2x - x^2$ is also a trinomial quadratic expression. To find its factors it can be written in the order $-x^2 + 2x + 4$.

When the trinomial quadratic expression $x^2 + 2xy - y^2$ is considered, it can be thought of as a quadratic expression in x or a quadratic expression in y. When it is considered as a quadratic expression in y, it is convenient to write it in the form $-y^2 + 2xy + x^2$.

For example, while $3x^2 - 2x - 5$, $a^2 + 2a + 8$, $y^2 + 2y - 5$ and $5 - 2x - 3x^2$ are trinomial quadratic expressions, the expressions $a + 2x + 3$ and $2x^3 + 3x^2 - 5x$ are not.

7.1 Factors of trinomial quadratic expressions

Let us recall how the product of two binomial expressions such as $x + 2$ and $x + 3$ is found.

$$\begin{aligned}(x+2)(x+3) &= x(x+3)+2(x+3)\\ &= x^2+\underbrace{3x+2x}+6\\ &= x^2+5x+6\end{aligned}$$

Since $x^2 + 5x + 6$ is obtained as the product of $x + 2$ and $x + 3$, we obtain that $x + 2$ and $x + 3$ are factors of $x^2 + 5x + 6$. $x^2 + 5x + 6$ is a trinomial qudratic expression. How do we obtain $x + 2$ and $x + 3$ as its factors? Let us examine the steps that were performed to obtain the product of the above two expressions, in the reverse order.

- The middle term $5x$ of the trinomial quadratic expression $x^2 + 5x + 6$, has been expressed as a sum of two terms, as $3x + 2x$
- The product of the terms $3x$ and $2x = 3x \times 2x = 6x^2$
- The product of the initial and final terms of the trinomial quadratic expression $x^2 + 5x + 6$ is $x^2 \times 6 = 6x^2$

We can come to the following conclusions based on the above observations.
The middle term should be written as the sum of two terms. The product of these two terms should be equal to the product of the initial and final terms of the trinomial expression.

As an example, let us factor $x^2 + 7x + 10$ accordingly. Here, $7x$ is the middle term. This should be written as the sum of two terms. Also, the product of these two terms should be equal to $10x^2$.

The product of the initial and final terms = $x^2 \times 10 = 10x^2$
The middle term = $7x$
Let us find the pair of terms which has $10x^2$ as its product and $7x$ as its sum. Let us consider the following table to do this. The two terms in the first column have been selected such that their product is $10x^2$.

Pair of terms	**Product**	**Sum**
$x,\ 10x$	$x \times 10x = 10x^2$	$x + 10x = 11x$
$2x,\ 5x$	$2x \times 5x = 10x^2$	$2x + 5x = 7x$
$(-x),\ (-10x)$	$(-x) \times (-10x) = 10x^2$	$(-x) + (-10x) = -11x$
$(-2x),\ (-5x)$	$(-2x) \times (-5x) = 10x^2$	$(-2x) + (-5x) = -7x$

It is clear from the above table that the middle term $7x$ should be written as $2x + 5x$. Now, let us find the factors of the given trinomial quadratic expression.

$$\begin{aligned} x^2 + 7x + 10 &= x^2 + 2x + 5x + 10 \\ &= x(x+2) + 5(x+2) \\ &= (x+2)(x+5) \end{aligned}$$

Therefore, the factors of $x^2 + 7x + 10$ are $x + 2$ and $x + 5$.

Let us see whether the factors are different if we write the middle term of $x^2 + 7x + 10$ as the sum $5x + 2x$, instead of $2x + 5x$.

$$\begin{aligned} x^2 + 7x + 10 &= x^2 + 5x + 2x + 10 \\ &= x(x+5) + 2(x+5) \\ &= (x+5)(x+2) \end{aligned}$$

We observe that the same factors are obtained. Therefore, the order in which we write the two terms that were selected has no effect on the factors. Accordingly, the factors can be found by writing $7x$ as either $2x + 5x$ or as $5x + 2x$.

Example 1

Factor $a^2 - 8a + 12$.

The product of the initial and final terms $= a^2 \times 12 = 12a^2$

The middle term $= -8a$

Find the two terms which have $12a^2$ as their product and $-8a$ as their sum.
The following table contains pairs of terms of which the product is $12a^2$. The pair which when added equals $-8a$ has been shaded.

Pair of terms	Product	Sum
a, $12a$	$a \times 12a = 12a^2$	$a + 12a = 13a$
$2a$, $6a$	$2a \times 6a = 12a^2$	$2a + 6a = 8a$
$3a$, $4a$	$3a \times 4a = 12a^2$	$3a + 4a = 7a$
$(-a)$, $(-12a)$	$(-a) \times (-12a) = 12a^2$	$(-a) + (-12a) = -13a$
$(-2a)$, $(-6a)$	$(-2a) \times (-6a) = 12a^2$	$(-2a) + (-6a) = -8a$
$(-3a)$, $(-4a)$	$(-3a) \times (-4a) = 12a^2$	$(-3a) + (-4a) = -7a$

That is, $-8a$ can be written as $-2a - 6a$

$$\begin{aligned} a^2 - 8a + 12 &= a^2 - 2a - 6a + 12 \\ &= a(a-2) - 6(a-2) \\ &= \underline{\underline{(a-2)(a-6)}} \end{aligned}$$

Note: A table has been used here only as an illustration. The middle term can be obtained as a sum mentally too.

Example 2

Factor $x^2 - 7x - 8$.

The product of the initial and final terms $= x^2 \times (-8) = -8x^2$

The middle term $= -7x$

The two terms which have $-8x^2$ as their product and $-7x$ as their sum are $+x$ and $-8x$.

Accordingly,

$$\begin{aligned} x^2 - 7x - 8 &= x^2 + x - 8x - 8 \\ &= x(x+1) - 8(x+1) \\ &= \underline{\underline{(x+1)(x-8)}} \end{aligned}$$

Now let us consider how an expression such as $-x^2 - x + 6$ which has a negative quadratic term is factored. The factors can be found by re-writing the expression as $6 - x - x^2$, with the quadratic term at the end. By considering the example given below, let us recognize the fact that the factors can be found using either form.

Example 3

Factor $-x^2 - x + 6$.
The product of the initial and final terms $= -6x^2$
The middle term $= -x$

$\therefore \quad -x$ should be written as $2x - 3x$.

$$\begin{aligned}
&-x^2 - x + 6\\
&= -x^2 + 2x - 3x + 6\\
&= x(-x+2) + 3(-x+2)\\
&= (-x+2)(x+3)\\
&= \underline{\underline{(2-x)(x+3)}}
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
&6 - x - x^2\\
&= 6 + 2x - 3x - x^2\\
&= 2(3+x) - x(3+x)\\
&= (3+x)(2-x)\\
&= \underline{\underline{(2-x)(x+3)}}
\end{aligned}$$

Example 4

Factor $a^2 - 4ab - 5b^2$.

We may consider this as a trinomial quadratic expression in a.
Then,
the product of the initial and final terms $= a^2 \times (-5b^2) = -5a^2b^2$
The middle term $= -4ab$

The two terms which have $-5a^2b^2$ as their product and $-4ab$ as their sum are ab and $-5ab$.

$$\begin{aligned}
a^2 - 4ab - 5b^2 &= a^2 + ab - 5ab - 5b^2\\
&= a(a+b) - 5b(a+b)\\
&= \underline{\underline{(a+b)(a-5b)}}
\end{aligned}$$

Note: This can also be considered as a trinomial quadratic expression in b and factored. Then too the above answer is obtained.

Accuracy of the factors of a trinomial quadratic expression

The errors made in simplification can be minimized by examing the accuracy of the factors of a trinomial quadratic expression once they have been found. For example, let us factor $x^2 + 3x - 40$.

$$\begin{aligned}
x^2 + 3x - 40 &= x^2 + 8x - 5x - 40\\
&= x(x+8) - 5(x+8)\\
&= \underline{\underline{(x+8)(x-5)}}
\end{aligned}$$

If the factors $x + 8$ and $x - 5$ are correct, by multiplying them together we should be able to get back our original expression. Let us find the product of $x + 8$ and $x - 5$

$$(x + 8)(x - 5) = x^2 - 5x + 8x - 40$$
$$= \underline{\underline{x^2 + 3x - 40}}$$

Since we have obtained our original expression $x^2 + 3x - 40$, the factors $x + 8$ and $x - 5$ are correct.

Exercise 7.1

1. Complete the following table.

Pair of algebraic terms	**Product**	**Sum**
$4x, x$	$4x^2$	$5x$
$2x, 7x$	....	....
$-5x, x$	....	....
$-3a, -7a$	....	....
$-p, -5p$	....	....
$2mn, -8mn$	....	....
....	$-4x^2$	$3x$
....	$-7x^2$	$6x$
....	$-10a^2$	$-3a$
....	$8p^2$	$6p$

2. Factor each of the following trinomial quadratic expressions.

A. **a.** $x^2 + 6x + 8$ **b.** $a^2 - 8a + 15$ **c.** $p^2 + 8p + 12$

d. $x^2 - 10x + 21$ **e.** $m^2 + 11m + 24$ **f.** $y^2 - 11y + 18$

g. $n^2 + 15n + 14$ **h.** $x^2 - 17x + 30$ **i.** $a^2 + 14a + 49$

j. $p^2 - 12p + 35$ **k.** $p^2 + 8p - 20$ **l.** $x^2 - 3x - 10$

m. $p^2 + p - 20$ **n.** $n^2 - 4n - 21$ **o.** $a^2 + 3a - 28$

p. $y^2 - 4y - 12$ **q.** $m^2 - 40 + 6m$ **r.** $5p + p^2 - 24$

s. $45 + x^2 - 14x$ **t.** $n^2 - 28 - 12n$

B. **a.** $10 - 3x - x^2$ **b.** $12 - p - p^2$ **c.** $12 - 4x - x^2$

d. $50 + 5x - x^2$ **e.** $18 + 7a - a^2$ **f.** $56 - y - y^2$

C. **a.** $a^2+7ab+10b^2$ **b.** $x^2+3xy+2y^2$

c. $p^2-7pq+12q^2$ **d.** $y^2+10ay+24a^2$

e. $a^2-10ab+21b^2$ **f.** $x^2-2xy-8y^2$

g. $p^2+pq-12q^2$ **h.** $y^2-3py-10p^2$

i. $a^2-ab-20b^2$ **j.** $x^2+6xy-40y^2$

3. A certain number is denoted by x. The product of the expression obtained by adding a certain number to x, and the expression obtained by subtracting a different number from x is given by the expression $x^2 + x - 56$.

(i) Find the factors of the given expression.

(ii) What is the number that has been added to the number denoted by x?

(iii) What is the number that has been subtracted from the number denoted by x?

7.2 The factors of trinomial quadratic expressions described further

So far we have considered only how the factors of trinomial quadratic expressions with the coefficient of x^2 equal to either 1 or -1 are found. Now let us consider how the factors are found when the coefficient of x^2 is some other integer. For example, let us consider the trinomial quadratic expression $3x^2 + 14x + 15$. This is of the form $ax^2 + bx + c$ with $a = 3$. In such cases too we can use the same method that we used above.

Example 1

Factor $3x^2 + 14x + 15$.

The product of the initial and final terms $= 45x^2$

The middle term $14x$ needs to be written as $5x + 9x$. (since $5x \times 9x = 45x^2$).

$$3x^2+14x+15$$
$$=3x^2+5x+9x+15$$
$$=x(3x+5)+3(3x+5)$$
$$=\underline{\underline{(3x+5)(x+3)}}$$

Example 2

Factor $6x^2 + x - 15$.

$6x^2 + x - 15$

$= 6x^2 + 10x - 9x - 15$

$= 2x(3x+5) - 3(3x+5)$

$= \underline{\underline{(3x+5)(2x-3)}}$

Example 3

Factor $2a^2 + 13ab - 7b^2$.

$2a^2 + 13ab - 7b^2$

$= 2a^2 - ab + 14ab - 7b^2$

$= a(2a-b) + 7b(2a-b)$

$= \underline{\underline{(2a-b)(a+7b)}}$

Example 4

Factor $x^2 + \frac{5}{2}x + 1$.

First let us write the algebraic expression with a common denominator.

$$x^2 + \frac{5}{2}x + 1 = \frac{2x^2 + 5x + 2}{2}$$

$$= \frac{1}{2}(2x^2 + 5x + 2)$$

Now let us find the factors of the quadratic expression within brackets.

$$2x^2 + 5x + 2 = 2x^2 + x + 4x + 2$$

$$= x(2x+1) + 2(2x+1)$$

$$= (2x+1)(x+2)$$

Therefore, $x^2 + \frac{5}{2}x + 1 = \frac{1}{2}(2x+1)(x+2)$

Exercise 7.2

1. Factor each of the following trinomial quadratic expressions.

A. **a.** $2x^2 + 3x + 1$ **b.** $5a^2 - 7a + 2$ **c.** $2x^2 - x - 1$

d. $4p^2 + 4p - 3$ **e.** $6x^2 + 3x - 3$ **f.** $2x^2 - 11xy + 15y^2$

g. $2y^2 - 5ya + 3a^2$ **h.** $2a^2 + 7ab + 6b^2$ **i.** $5p^2 - 9pq - 2q^2$

j. $2m^2 + 3mn - 2n^2$ **k.** $x^2y^2 + 10xy + 16$ **l.** $2x^3 - x^2y - 3xy^2$

2. Find the value of each of the following numerical expressions using the knowledge on the factors of trinomial quadratic expressions.

a. $8^2 + 7 \times 8 + 10$ **b.** $93^2 + 3 \times 93 - 28$

c. $27^2 - 4 \times 27 - 21$ **d.** $54^2 + 2 \times 54 - 24$

7.3 The factors of expressions which are in the form of the difference of two squares

Consider the product of the two binomial expressions $(x - y)$ and $(x + y)$

$$(x - y)(x + y) = x^2 + xy - xy - y^2$$
$$= x^2 - y^2$$

We have obtained $x^2 - y^2$ which is a difference of two squares. We observe from the above expansion that the factors of $x^2 - y^2$ are $x - y$ and $x + y$. Further, $x^2 - y^2$ can be considered as a quadratic expression in x and factored. By taking its middle term to be 0, we can write it in the form of a trinomial quadratic expression as $x^2 + 0 - y^2$. Let us factor this expression.

Product of the initial and final terms $= - x^2y^2$

Middle term $= 0$

Accordingly, the pair of terms with product $- x^2y^2$ and sum 0 is $- xy$ and xy.

$$\begin{aligned} x^2 + 0 - y^2 &= x^2 - xy + xy - y^2 \\ &= x(x - y) + y(x - y) \\ &= \underline{\underline{(x - y)(x + y)}} \end{aligned}$$

Therefore, by this too we obtain $x^2 - y^2 = (x - y)(x + y)$.

Let us consider the following examples of the factorization of expressions which are the difference of two squares.

Example 1

Factor (i) $x^2 - 4$ (ii) $4x^2 - 9$ (iii) $25a^2 - 16b^2$

(i)

$$\begin{aligned} & x^2 - 4 \\ &= x^2 - 2^2 \\ &= \underline{\underline{(x - 2)(x + 2)}} \end{aligned}$$

(ii)

$$\begin{aligned} & 4x^2 - 9 \\ &= (2x)^2 - 3^2 \\ &= \underline{\underline{(2x - 3)(2x + 3)}} \end{aligned}$$

(iii)

$$\begin{aligned} & 25a^2 - 16b^2 \\ &= (5a)^2 - (4b)^2 \\ &= \underline{\underline{(5a - 4b)(5a + 4b)}} \end{aligned}$$

Do the following exercise after studying the above examples.

Exercise 7.3

1. Fill in the blanks.

(i) $x^2 - 36$
$= x^2 - ....^2$
$= (x-6)(x+6)$

(ii) $9 - y^2$
$= .... - ....$
$= (.......)(.......)$

(iii) $25x^2 - 4y^2$
$= (.......)^2 - (.......)^2$
$= (.......)(.......)$

(iv) $2a^2 - 8b^2$
$= 2(.............)$
$= 2(a^2 - (......)^2)$
$= 2(......)(......)$

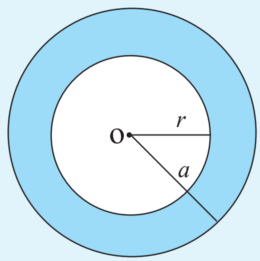
(v) $3p^2 - 27q^2$
$= 3(..... - .....)$
$= 3\left[(.....)^2 - (.....)^2\right]$
$= ......(......)(......)$

(vi) $a^2b^2 - 1$
$= (ab)^2 - 1$
$= (....-....)(....+.....)$

2. Factor the following algebraic expressions.

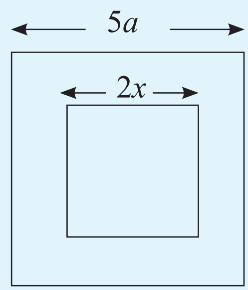
a. $y^2 - 81$ **b.** $16 - b^2$ **c.** $100 - n^2$ **d.** $m^2n^2 - 1$
e. $16a^2 - b^2$ **f.** $4x^2 - 25$ **g.** $9p^2 - 4q^2$ **h.** $400 - 4n^2$
i. $8x^2 - 2$ **j.** $4x^2y^2 - 9y^2$

3. The figure depicts two concentric circles of centre O. The radius of the smaller circle is r and the radius of the larger circle is a.



(i) Express the area of the smaller circle in terms of π and r.
(ii) Express the area of the larger circle in terms of π and a.
(iii) Write an expression for the area of the shaded region in terms of π, r and a and then represent it as a product of two factors.

4. The figure depicts two squares of side length $5a$ units and $2x$ units respectively.



(i) Express the area of the smaller square in terms of x.
(ii) Express the area of the larger square in terms of a.
(iii) Show that the area of the larger square is greater than the area of the smaller square by $(5a + 2x)(5a - 2x)$ square units.

7.4 Factors of the difference of two squares described further

There are many algebraic expressions that can be factored by considering them as the difference of two squares. Given below are two such expressions.

Example 1

Factor each of the following algebric expressions.

(i) $(x+2)^2 - y^2$ (ii) $(a-2)^2 - (a+5)^2$

(i) $(x+2)^2 - y^2$

$= [(x+2)-y]\ [(x+2)+y]$

$= \underline{\underline{(x+2-y)(x+2+y)}}$

(ii) $(a-2)^2 - (a+5)^2$

$= [(a-2)-(a+5)]\ [(a-2)+(a+5)]$

$= [a-2-a-5][a-2+a+5]$

$= \underline{\underline{-7(2a+3)}}$

Exercise 7.4

Factor the following expressions.

a. $(x+1)^2 - 4$ **b.** $(y-2)^2 - 9$ **c.** $(2a+3)^2 - 49$

d. $(4x-3y)^2 - 25$ **e.** $(2p+3)^2 - 4q^2$ **f.** $25-(x+3)^2$

g. $4-(a-2)^2$ **h.** $16-(m+2)^2$ **i.** $(m+2)^2-(m+1)^2$

j. $(2x+3)^2-(x-2)^2$

Miscellaneous Exercise

1. Factor the following expressions

a. $(x-y)^2 - 4a^2b^2$ **b.** $x^2y^2 + 10xy + 16$

c. $p^2q^2 - pq - 20$ **d.** $2x^3 - x^2y - 3xy^2$

e. $6x^2 - 2x - 4$ **f.** $(x+1)^2 - (x-3)^2$

g. $x(x+5) - 14$ **h.** $(2x-1)^2 - 4$

2. Factor the following expressions. (Hint: Take $x^2 = y$)

a. $x^4 + 5x^2 + 6$ **b.** $x^4 - 16$

c. $2x^4 + 14x^2 + 24$ **d.** $1 - 81x^4$