4 Binomial Expressions

By studying this lesson you will be able to

- multiply two binomial expressions
- expand the square of a binomial expression

Do the following exercise to recall the facts you have learnt earlier about simplifying algebraic expressions.

Review Exercise

1. Simplify the following algebraic expressions.

a. $2 \times 3a$ **b.** $4 \times (-2x)$ **c.** $(-3) \times 2x$
d. $2x \times 3y$ **e.** $3a \times (-5b)$ **f.** $(-2m) \times 4n$
g. $(-4p) \times (-2q)$ **h.** $3x \times 5x$ **i.** $(-5a) \times 3a$

2. Expand the following algebraic expressions.

a. $2(x+1)$ **b.** $3(b+3)$ **c.** $4(y-2)$
d. $-3(a+2)$ **e.** $-2(x-2)$ **f.** $x(2x+3)$
g. $2y(y+1)$ **h.** $-2x(4x+1)$ **i.** $-3b(a-b)$
j. $2(a-b-3c)$

3. Expand and simplify the following algebraic expressions.

(a) (i) $x(x+2)+2(x+2)$ **(ii)** $y(y+3)+3(y-2)$
(iii) $x(x+1)-3(x-1)$ **(iv)** $m(m-3n)-n(m-3n)$

(b) (i) $(x+5)(x+8)$ **(ii)** $(7+a)(3+a)$ **(iii)** $(x-5)(x+8)$
(iv) $(x+5)(x-8)$ **(v)** $(2+m)(3-m)$ **(vi)** $(x-5)(x-8)$

4.1 Product of two binomial expressions

What you simplified in 3(*b*) above is the product of two binomial expressions of the form $x + a$. In this lesson, we will learn about expanding the product of two binomial expressions of the form $ax + by$. Here, ax and by are the two terms of the binomial expression $ax + by$.

Example 1 Expand and simplify $(3x + 2)(2x + 3)$.

$(3x+2)(2x+3)$

$= 3x(2x+3)+2(2x+3)$

$= 6x^2+9x+4x+6$

$= \underline{\underline{6x^2+13x+6}}$

or

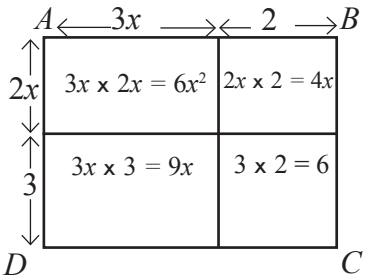
$(3x+2)(2x+3)$

$= (3x+2)\times 2x+(3x+2)\times 3$

$= 6x^2+4x+9x+6$

$= \underline{\underline{6x^2+13x+6}}$

The result that was obtained above can also be illustrated using the areas of rectangles. (All the measurements below are assumed to have been given in the same units).



For the rectangle $ABCD$,

length of $AB = 3x + 2$

length of $AD = 2x + 3$

Area $= (3x + 2)(2x + 3)$ ——— ①

According to the figure,

Area of the rectangle $ABCD$ = The sum of the areas of the four smaller rectangles

$= 6x^2 + 9x + 4x + 6$

$= 6x^2 + 13x + 6$ ——— ②

It is clear according to (1) and (2) that,

$(3x + 2)(2x + 3) = 6x^2 + 13x + 6$

Now study how the binomial expressions of different types given in the following examples have been expanded and simplified.

Example 2

$(3x-2)(2x+5)$

$(3x-2)(2x+5)$

$= 3x(2x+5)-2(2x+5)$

$= 6x^2+15x-4x-10$

$= \underline{\underline{6x^2+11x-10}}$

Example 3

$(2x+y)(x+3y)$

$(2x+y)(x+3y)$

$= 2x(x+3y)+y(x+3y)$

$= 2x^2+6xy+xy+3y^2$

$= \underline{\underline{2x^2+7xy+3y^2}}$

Example 4

$(3x+2y)(3x-2y)$

$(3x+2y)(3x-2y)$

$= 3x(3x-2y)+2y(3x-2y)$

$= 9x^2-6xy+6xy-4y^2$

$= \underline{\underline{9x^2-4y^2}}$

Example 5

$(5a-2b)(2a-3b)$

$(5a-2b)(2a-3b)$

$= 5a(2a-3b)-2b(2a-3b)$

$= 10a^2-15ab-4ab+6b^2$

$= \underline{\underline{10a^2-19ab+6b^2}}$

Example 6

$(a+b)(\frac{1}{3}a-\frac{1}{4}b)$

$(a+b)(\frac{1}{3}a-\frac{1}{4}b)$

$= a(\frac{1}{3}a-\frac{1}{4}b)+b(\frac{1}{3}a-\frac{1}{4}b)$

$= \frac{1}{3}a^2-\frac{1}{4}ab+\frac{1}{3}ab-\frac{1}{4}b^2$

$= \underline{\underline{\frac{1}{3}a^2+\frac{1}{12}ab-\frac{1}{4}b^2}}$

Exercise 4.1

1. Expand and simplify the following binomial expressions.

a. $(x+2)(x+2)$ **b.** $(x-3)(x-3)$ **c.** $(2x+3)(x+2)$

d. $(2p-5)(p-3)$ **e.** $(3x-1)(3x+1)$ **f.** $(-3x+2)(2x-3y)$

g. $(2a+b)(3a+2b)$ **h.** $(3x-5y)(4x+3y)$ **i.** $(-3p+4q)(3p-2q)$

j. $(-7k-5l)(3k+4l)$ **k.** $(4m-3n)(4m-3n)$ **l.** $(5x-2y)(5x-2y)$

m. $\left(\frac{1}{2}x+y\right)(2x+3y)$ **n.** $\left(\frac{1}{3}p+\frac{1}{2}q\right)\left(\frac{2}{3}p-\frac{3}{4}q\right)$ **o.** $(3x+4y)(5a+3b)$

2. If the length of a rectangular shaped field is $(2a + 7)$ m and the breadth is $(2a - 3)$ m, determine the area of the field in terms of a.

3. Piyumi made a square - shaped flower bed. Her sister made a rectagular flower bed. The length of the sister's bed was 3 metres more than of Piyumi's and the width was 2 metres less than of Piyumi's. Taking the length of one side of Piyumi's bed as x, write the length and width of the sister's bed in terms of x and then express its area in the form $Ax^2 + Bx + C$.

4. A child bought a mandarins which were priced at x rupees each. He then decides to buy three times that number of apples. The price of an apple is twice the price of a mandarin.

(i) Write an expression in terms of a and x for the cost of the apples.

The fruit seller states that if the number of apples that are bought is increased by 5 fruits, he can reduce the price of each apple that is bought by 3 rupees. Accordingly, the child decides to buy 5 more apples.

(ii) If he does this, write down the number of apples he buys in terms of a.

(iii) Write down the price of an apple in terms of x.
(iv) Write down the cost of the apples in terms of a and x.
(v) Expand and simplify the binomial expression in (iv) above.

4.2 Squares of binomial expressions

Let us turn our attention again to the following expressions which you expanded in the above exercise. Do you notice that the two binomial expressions to be multiplied are equal?

$(x + 2)\ (x + 2)$, $(x - 3)\ (x - 3)$, $(5x - 2y)\ (5x - 2y)$

In the same manner that we write $x \times x = x^2$, we can write $(x + 2)(x + 2) = (x + 2)^2$.

Similarly,

$$(x - 3)\ (x - 3) = (x - 3)^2$$
$$(5x - 2y)\ (5x - 2y) = (5x - 2y)^2$$

Each of these expressions written in the form $(x + 2)^2$, $(x - 3)^2$ and $(5x - 2y)^2$ is called the square of the respective binomial expression.

The same method that was used above to simplify the product of two binomial expressions can be used to expand such squares of binomial expressions too.

Example 1

Write $(x + 2)^2$ as a product of two binomial expressions and expand it.

$$\begin{aligned}(x+2)^2 &= (x+2)\,(x+2)\\ &= x\,(x+2) + 2\,(x+2)\\ &= x^2 + 2x + 2x + 4\\ &= \underline{\underline{x^2 + 4x + 4}}\end{aligned}$$

Simplifying the square of a binomial expression can be done using another method too.

Now let us consider how the squares of binomial expressions of the form $(a + b)$ and $(a - b)$ are expanded.

$$\begin{aligned}(a+b)^2 &= (a+b)\,(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + ab + ab + b^2\\ &= \underline{\underline{a^2 + 2ab + b^2}}\end{aligned}$$

It is important to remember this as a formula.

$$(a+b)^2 = a^2 + 2ab + b^2$$

Square of the first term — Two times the product of the first and second terms — Square of the second term

Now let us consider the expansion of $(a-b)^2$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - ab - ab + b^2\\ &= \underline{\underline{a^2 - 2ab + b^2}}\end{aligned}$$

That is, $\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

Note : The expansion of $(a-b)^2$ can also be obtained by substituting $-b$ for b in the expansion of $(a+b)^2$

i.e., $(a+(-b))^2 = a^2 + 2(a)(-b) + (-b)^2 = a^2 - 2ab + b^2$

Also,

$$(-a+b)^2 = (-a)^2 + 2(-a)b + b^2 = a^2 - 2ab + b^2$$

$$(-a-b)^2 = (-a)^2 + 2(-a)(-b) + (-b)^2 = a^2 + 2ab + b^2$$

You may have observed that the expansions of $(a+b)^2$ and $(-a-b)^2$ are equal to each other and that the expansions of $(a-b)^2$ and $(-a+b)^2$ too are equal to each other.

Now study how the following squares of binomial expressions have been expanded.

Example 2

$$\begin{aligned}(x+3)^2 &= x^2 + 2\times x\times 3 + 3^2\\ &= \underline{\underline{x^2 + 6x + 9}}\end{aligned}$$

Example 3

$$\begin{aligned}(y-2)^2 &= y^2 - 2\times y\times 2 + 2^2\\ &= \underline{\underline{y^2 - 4y + 4}}\end{aligned}$$

Example 4

$$\begin{aligned}(3x+5y)^2 &= (3x)^2 + 2\times 3x\times 5y + (5y)^2\\ &= \underline{\underline{9x^2 + 30xy + 25y^2}}\end{aligned}$$

Example 5

$$\begin{aligned}(3a-2b)^2 &= (3a)^2 - 2\times 3a\times(2b) + (2b)^2\\ &= \underline{\underline{9a^2 - 12ab + 4b^2}}\end{aligned}$$

Example 6

$$(-y+5)^2 = (-y)^2 - 2\times(y)\times 5 + 5^2$$
$$= \underline{\underline{y^2 - 10y + 25}}$$

Example 7

$$(-2x-3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$$
$$= \underline{\underline{4x^2 + 12xy + 9y^2}}$$

Let us consider how this method is used to find numerical values.

Example 8

Find the value of 105^2.

$105^2 = (100 + 5)^2$
$= 100^2 + 2 \times 100 \times 5 + 5^2$
$= 10000 + 1000 + 25$
$= \underline{\underline{11025}}$

Example 9

Find the value of 99^2

$99^2 = (100 - 1)^2$
$= 100^2 - 2 \times 100 \times 1 + 1^2$
$= 10000 - 200 + 1$
$= \underline{\underline{9801}}$

Example 10

Verify $(x + y)^2 = x^2 + 2xy + y^2$ for the values $x = 5$ and $y = 2$.

l.h.s
$(x+y)^2$
$= (5+2)^2$
$= 7^2$
$= \underline{\underline{49}}$

r.h.s. $x^2 + 2xy + y^2$
$= 5^2 + 2\times 5\times 2 + 2^2$
$= 25 + 20 + 4$
$= \underline{\underline{49}}$

$\therefore$ l.h.s = r.h.s.

Exercise 4.2

1. For each expression in column *A* which is a square of a binomial expression, select and join the corresponding expansion from Column *B*.

Column *A*	**Column *B***
a. $(x + 5)^2$	$4x^2 + 4xy + y^2$
b. $(x - 5)^2$	$4y^2 + 4xy + x^2$
c. $(2x + 5)^2$	$x^2 - 10x + 25$
d. $(2x + y)^2$	$4x^2 - 4xy + y^2$
e. $(-2x + 5)^2$	$x^2 - 4xy + 4y^2$
f. $(x - 2y)^2$	$4x^2 - 12xy + 9y^2$
g. $(-2x + y)^2$	$4x^2 + 20x + 25$
h. $(2x + 3y)^2$	$4x^2 + 12xy + 9y^2$
i. $(2x - 3y)^2$	$x^2 + 10x + 25$
j. $(-2y - x)^2$	$4x^2 - 20x + 25$

2. Expand each of the following squares of binomial expressions.

a. $(x + 2)^2$ **b.** $(a + 3)^2$ **c.** $(p - 3)^2$ **d.** $(y - 1)^2$

e. $(2a + 3)^2$ **f.** $(3b + 2)^2$ **g.** $(3x - 1)^2$ **h.** $(4m - 5)^2$

i. $(3p + 4q)^2$ **j.** $(5m - 3n)^2$ **k.** $(-2y + 5)^2$ **l.** $(3a - 5b)^2$

m. $(-3m + n)^2$ **n.** $(-5m - 6n)^2$

3. Write down the term suitable for the blank space in each of the following expressions.

a. $(x + 3)^2 = x^2 + 6x +$ ___ **b.** $(y + 2)^2 = y^2 +$ ___ $+ 4$

c. $(m - 5)^2 = m^2 - 10m +$ ___ **d.** $(a +$ __ $)^2 = a^2 + 8a + 16$

e. (___ $+ b)^2 = 25 + 10b + b^2$ **f.** (___ $- 7)^2 = x^2 - 14x + 49$

g. $(-3 +$ ___ $)^2 =$ ___ $- 6x + x^2$ **h.** (___ $- x)^2 = +16 - 8x + x^2$

4. Write each of the following as a square of a binomial expression and find the value.

(i) 21^2 (ii) 102^2 (iii) 17^2 (iv) 98^2 (v) 9.9^2

5. If the length of a side of a square shaped room is given as $(2a + 3b)$ meters, write down an expression for the area of the room in terms of a and b and expand it.

6. Verify the following for the values $a = 2$ and $b = 3$.

(i) $(-a + b)^2 = a^2 - 2ab + b^2$

(ii) $(-a - b)^2 = a^2 + 2ab + b^2$

Miscellaneous Exercise

1. Verify that $(2x + 3y)(x + y) = 2x^2 + 5xy + 3y^2$ for each of the following cases.

(i) $x = 3, y = 2$ (ii) $x = 5, y = 0$

(iii) $x = 1, y = 1$ (iv) $x = -1, y = -2$

2. Write each of the following squares of binomial expressions with fractional coefficiants as a product of two binomial expressions and simplify it.

(i) $(\frac{1}{2}x + y)^2$ (ii) $(\frac{1}{3}a - b)^2$ (iii) $(\frac{1}{4}m - \frac{2}{3}n)^2$

3. Fill in the blanks

(i) $(x + \_)^2 = x^2 + 6x + \_$ (ii) $(y + \_)^2 = y^2 + 8y + \_$

(iii) $(\_ + 5)^2 = x^2 + \_ + 25$ (iv) $(\_ + y)^2 = x^2 + \_ + y^2$

4. For each of the following expressions, determine the term that should be added so that it can be written as a square of a binomial expression, add the relevant term to the given expression and then write it as a square of a binomial expression.

(i) $x^2 + 6x$ (ii) $y^2 + 8y$ (iii) $m^2 + 10m$

(iv) $a^2 - 4a$ (v) $x^2 + 4xy$ (vi) $p^2 - 12pq$

5. Find the value of $x^2 + y^2$ when $x + y = 5$ and $xy = 6$.

6. Find the value of $a^2 + b^2$ when $a - b = 3$ and $ab = 28$.

7. Find the value of $x + y$ when $x^2 + y^2 = 25$ and $xy = 12$.

8. If $(x + k)^2 = x^2 + 6x + q$, find the values of k and q.

9. Find the value of $t^2 + \frac{1}{t^2}$ when $t + \frac{1}{t} = 2$.