Royal College Colombo 07
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General Certificate of Education (Adv. Level) Examination, 2010
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Grade 13 – Final Term Test June 2010
13 වන ශ්‍රේණිය අවසාන වාර පරීක්ෂණය 2010 ජූනි

Time – 3 Hours

Combined Mathematics I

Answer 6 questions only.

(01) (i) $f(x) = ax^2 + bx + c$ is a quadratic expression where $a < 0$.

If $b^2 - 4ac > 0$ then

showthat there exist a constant $\lambda (> 0)$ such that

$f(x) \leq \lambda$ for all $x \in \Re$

Hence or other wise

showthat the quadratic equation

$x^2 + x - k(x - \ell)(x - m) + 1 = 0$ has real and distinct roots

where ℓ, m, k are constant and $k > 1$.

and show also that ℓ and m are two numbers belong to the interval of above two roots.

If n is the arithmetic mean of two roots then find the value of k in terms of ℓ, m and n.

(ii) Find the possible values of x for which,

$$\log_3(2x+5) + \frac{1}{\log_{(x+1)} 3} = 2$$

(iii) The polynomial $f(x) \equiv x^3 - 3x^2 + Px + 8$ is divide by $x - 2$, the remainder is 2,

Find the value of P.

hence or otherwise

If $g(x) \equiv x^2 - x - 4$ and $H(x) = f(x) + g(x)$

Solve the equation $H(x) = 0$

(02)(i) In how many ways can 6 gents and 6 ladies be arranged at a round table, if the two particular ladies Miss X and Miss Y refuse to sit next to Mr. Z. All men being separated.

(ii) Write down the expansion of $\left(x^2+\frac{a}{x}\right)^{2n+1}$

when $n \in Z^+$ and a is non zero constant.

Show that there exist a independent term of x in the above expansion

when $(n+2)$, is exactly divisible by 3.

When $n=13$ this term is ${}^{27}C_9(27)^6$ find the values of a.

Also find the greatest coefficient of the expansion when $a > 0$.

(03) (i) Find the set of values of x for which

$\frac{2x+5}{|x-1|} \le 3$ where $x \in \Re$.

(ii) Prove by mathematical induction

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+.........+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ Where $n \in Z^+$

(iii) Write down the r^{th} term U_r of the series,

$\frac{1}{1.3.5}+\frac{1}{2.4.6}+\frac{1}{3.5.7}+......$

Express U_r as a partial fractions.

Hence or otherwise

Find f(r) such that $U_r = f(r+1) - f(r)$

Find $S_n = \sum_{r=1}^{n} U_r$

Is this series convergent, justify your answer.

Show that $\frac{8}{15} \le 8S_n \le \frac{11}{12}$

(04) (i) Let $f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$

If $z = z_0$ is a root of the equation $f(z) = 0$,

then showthat $z = \overline{z_0}$ is also a root of the equation $f(z) = 0$

If $x = 2i - 1$ is a root of the equation $4x^4 + 8x^3 + 11x^2 - 18x - 45 = 0$,

then find the remaining roots of the equation.

(ii) Describe the locus of point z in the argend plane when z is given by,

(a) $|z - 3 - 4i| = 4$ (b) $\arg(z - 3 - 4i) = \frac{\pi}{3}$

Find the equations of locus of z.

If, (i) $3 \le |z - 3 - 4i| \le 4$ (ii) $0 \le \arg(z - 3 - 4i) \le \frac{\pi}{3}$

then shade the common region that z is satisfied above inequalities (i) and (ii).

Find the maximum and minimum of $|z|$ in this region.

(05) (i) When $x > 0$, show that $\frac{d(\ln x)}{dx} = \frac{1}{x}$.

If $y = \frac{x^2}{2} + \frac{1}{2} x\sqrt{x^2 + 1} + \ell n\sqrt{x + \sqrt{x^2 + 1}}$ then,

Showthat $2y = x\frac{dy}{dx} + \ln\left(\frac{dy}{dx}\right)$.

(ii) If (x,y) is any point on a circle of radius r and centre (a, b).

Then showthat $\frac{d^2 y}{dx^2} = -\frac{r^2}{(y - b)^3}$

deduce that $\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d^3 y}{dx^3} = 3\left(\frac{dy}{dx}\right)\left(\frac{d^2 y}{dx^2}\right)^2$.

(iii) a curve is given by $x = aCos^3\theta$ and $y = aSin^3\theta$ where $0 < \theta < \frac{\pi}{2}$ and $a > 0$.

Find the equations of tangent and normal to the curve at "θ".

If p and q are the distances from origin O to the above tangent and normal.

Then show that $4p^2+q^2$ is independent of θ.

(06) (i) If $\lambda(ax^2+1)+(\mu x+\upsilon)\frac{d}{dx}(ax^2+1)\equiv x^2+1,(a\neq 0)$ then find the constant λ,μ and υ.

Hence find $\int \frac{x^2+1}{(ax^2+1)^2}dx$.

(ii) If $f(x)=\frac{Sin2x}{Cos^4x+Sin^4x}$ showthat $f(\pi+x)=f(x)$.

By using above result showthat $\int_0^{\frac{\pi}{4}} f(x)dx=\int_{\pi}^{\frac{5\pi}{4}} f(x)dx$.

Hence showthat $\int_{\pi}^{\frac{5\pi}{4}} \frac{Sin2x}{Cos^4x+Sin^4x}dx=\frac{\pi}{4}$.

(iii) By suitable substitution,

Showthat $\int_0^{\frac{\pi}{4}} \frac{\sqrt{Cos2\theta}}{Cos\theta}d\theta=\frac{\pi}{2}\left(\sqrt{2}-1\right)$.

(07) If two straight lines $U_1=a_1x+b_1y+c_1=0$ and $U_2=a_2x+b_2y+c_2=0$ intersect at point G, then prove that the equation of straight line through G is $U_1+\lambda U_2=0$ where λ is a parameter.

The straight line intersects above lines U_1 and U_2 at points A and B respectively and F is a point on AB such that $\frac{AF}{FB}=k$.

Showthat the equation of the straight line FG is given by $(ba_2-ab_2)U_1+k(ba_1-ab_1)U_2=0$ where k is a parameter.

The sides *AB*, *BC*, *CA* of a Δ *ABC* lie along the lines $y-x-1=0$, $2y+x-5=0$ and $y-3x+1=0$ respectively. *D* is point on *AC*, and *E* is a point on *AB* such that $\frac{AD}{DC}=4$ and $\frac{AE}{EB}=\frac{2}{3}$ respectively. By using above result find the equations of lines *BD* and *CE*.

Hence or othewise find the equation of line *AP*, where *P* is the point of intersection of lines *BD* and *CE*. If *AP* and *BC* meet at pint *Q*. Hence deduce the ratio $\frac{BQ}{QC}$.

(08)(i) Find the equation of the tangent to the circle $S \equiv x^2 + y^2 + 2xy + 2fy + c = 0$ at the point (x_1, y_1) which is on the circle *S*.

Hence showthat the equation of chord of contact to the circle $x^2 + y^2 - 2ay + 2a - 1 = 0$ which is drawn from the point $A \equiv (\lambda, 0)$ is $\lambda x - ay + 2a - 1 = 0$ (Where $a > 1$)

If this chord is BC and BC subtend angle 90^0 in the origin O then

showthat $a \geq 2 + \sqrt{2}$.

If a = 4 then showthat the distance between A and O is 2 units.

(ii) Two inclined straight lines $y = mx + c$ and $y = m^/ x + c^/$ meet Ox and Oy at 4 distinct points. If these 4 points are on a circle, without finding the equation of the circle. Showthat $mm^/ = 1$.

Prove that the centre of the circle is $\left[-\frac{(c^/ m + m^/ c)}{2}, \frac{c + c^/}{2} \right]$

and the radius is $\frac{\sqrt{c^2 (m^{/2} + 1) + c^{/2} (m^2 + 1)}}{2}$.

(09) (i) If $\tan x = \frac{1 - Cosy}{Siny}$

then showthat $y = 2x$.

Deduce that $\tan 7\frac{1}{2}^0 = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

(ii) Solve the $\left(Cos\frac{\theta}{4} - 2Sin\theta \right) Sin\theta + \left(1 + Sin\frac{\theta}{4} - 2Cos\theta \right) Cos\theta = 0$

(iii) Express sine rule for any Δ ABC

If in a Δ ABC, Cos A. Cos B + Sin A. Sin B. Sin C = 1 then

Prove that A = B and a : b : c = 1: 1: $\sqrt{2}$.

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Time – 3 Hours

Combined Mathematics II

Answer 6 questions only.

(1)(a) Points A and B are on a rough plane which is inclined 30^0 to the horizontal. AB = a and A is below B. A particle P is projected along the plane from A towards B with speed U and simultaneously another particle Q is projected from B towards A with speed U. The coefficient of friction between P and Q with the plane are $\frac{1}{2\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ respectively.

By considering the distance travelled by each particle before coming to rest, draw v-t graphs for the motion of P and Q in the same axes.

By using the graphs showthat $U^2 \geq \frac{3ag}{5}$.

If $U^2 = \frac{8ag}{3}$ then showthat the collision occurs after a time $\frac{2}{5}\sqrt{\frac{2a}{3g}}$ and the distance from A is $\frac{37a}{75}$.

(b) Two students A and B stand on the two points which are on a horizontal ground d meters apart. A is projected a stone P with the velocity U ms^{-1} vertically upwards. When P reached the highest point of its path, B is projected another stone Q with the velocity V ms^{-1} to collied with the stone P.

Find the velocity of Q relative to P when P is in the highest point, also find the acceleration of Q relative to P.

Hence (i) Find the direction of projection velocity of Q.

(ii) Showthat the condition that the stones P and Q collied each other before P hits the ground is $V \geq \frac{\sqrt{U^4 + 4g^2d^2}}{2U}$

(2)(a) A smooth wedge is placed on a rough horizontal plane. A particle is placed on a smooth plane of the wedge inclined at an angle 60^0 to the horizontal. The mass of the wedge is n times the mass of the particle and the coefficient of friction between the wedge and the plane is $\frac{1}{3\sqrt{3}}$,

i) If $n \geq 2$ then showthat the wedge is at rest.

ii) If $n < 2$ the wedge is in motion then showthat its acceleration is $\frac{(2-n)g}{\sqrt{3}(3n+2)}$

(b) Two small smooth spheres A, B of equal radii and of masses m, λm respectively. The sphere A is projected with a velocity U towards B. The coefficient of restitution between the spheres is $\frac{2}{3}$. After the collision find the velocities of the spheres A and B. After being struck by A the sphere B goes on to strike the wall. The coefficient of restitution between B and the wall is $\frac{2}{3}$. If there is no second collision between A and B then showthat $\lambda \geq \frac{19}{6}$.

When $\lambda = 3$, find the kinetic energy lost in the system as a fraction of the original kinetic energy.

(3) A uniform smooth hemispherical solid block of mass 3m and radius a is free to slide with its base on a smooth horizontal table. Centre of the circular base is O. A is the highest point on it. A particle of mass m is placed on the block at B such that the angle AOB = $\frac{\pi}{3}$ and then released from rest. After a time t the particle is on the block at P such that the angle AOP = $\theta\left(>\frac{\pi}{3}\right)$. The motion happens along the vertical plane which goes through the centre O of the block. If $\dot{\theta}$ is the angular velocity of the particle relative to the block then by suing the principle of conservation of energy and momentum showthat $\dot{\theta}^2 = \frac{4g(1-2Cos\theta)}{a(3+Sin^2\theta)}$.

When the particle leaves the block at an angle α with the upward vertical then showthat $Cos^3\alpha = 4(3Cos\alpha - 1)$

(4) Two particles A and B of masses m and $3m$ respectively are attached to the ends of a light elastic string to natural length ℓ and modulus of elasticity mg and are kept on a smooth horizontal plane such that $AB = \ell$. The particle A is projected towards $\overrightarrow{BA}$ with a velocity V . At time t , $AB = \ell + x$ and y is the distance travelled by the particle B.

Showthat $\ddot{x} + n^2 x = 0$, where $n^2 = \dfrac{4g}{3\ell}$.

Given that $x = A\ Cos\ nt + B\ Sin\ nt$ is the solution of the above equation, find the values of the constants A and B.

Hence showthat $nx = V\ Sin\ nt$ and also showthat $\ddot{y} = \dfrac{V}{6}\sqrt{\dfrac{3g}{\ell}}\ Sin\ nt$.

Given that $4ny = V(nt - Sin\ nt)$ is the solution of the above equation

showthat $y = \dfrac{\pi V}{8}\sqrt{\dfrac{3\ell}{g}}$ when AB becomes ℓ .

(5)(a) A rectangle ABCD has AB = a and AD = 2a and M is the mid point of AD. Forces P, 2P, 4P, 6P, $3\sqrt{2}$ P and $\sqrt{5}$ P act along CB, DA, BA, CD, MB, DB respectively, the direction of the forces being indicated by the order of the letters.

Reduce the system to a single force acting through A and a couple.

State the magnitude and direction of the force, and showthat the couple has moment 6aP. Where does the resultant of the system cut AD?

Find two parallel forces through B and D which are together equivalent to the system.

(b) A hollow circular cylinder of thin material has diameter $2a$, length $2a\ tan\ \alpha$, and weight W . It is open at both ends and stands with one end resting on a smooth horizontal table. A uniform rod AB of weight w and length $2\ell\ (>2a\ Sec\ \alpha)$ is placed inside the cylinder to rest against the upper rim touches at C with its lower end in contact with both the table an the wall of the cylinder at A. All the surfaces are smooth.

i) Showthat the reaction at C is $\dfrac{w\ell Cos^2\alpha}{2a}$

ii) Find the vertical component of the reaction at A and showthat $2a > \ell Cos^3\alpha$.

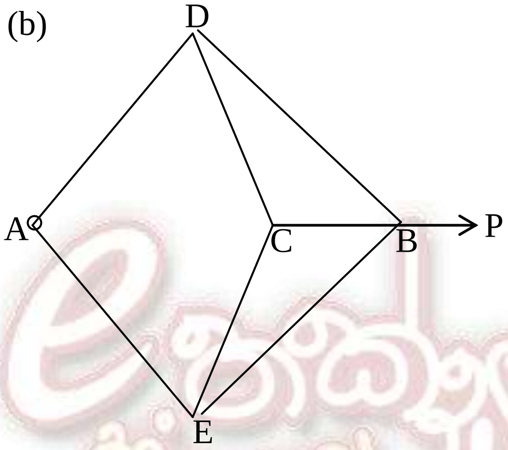
iii) Find the vertical component of the reaction at A on the cylinder and showthat $Wa > w\ell\ Sin^2\alpha\ Cos\ \alpha$.

(6)(a) AB, BC, CD and DE are four uniform rods of equal weights and equal length $2a$. They are freely jointed together by smooth hinges at B, C and D and stand in equilibrium in a vertical plane with the ends A and E on a rough horizontal plane, in the form of a symmetrical arch. The coefficient of friction between the plane and the rods is $\frac{1}{4}$.

Prove that the maximum span AE of the arch is $\frac{2a}{5}\left(\sqrt{10}+5\sqrt{2}\right)$ and find the corresponding height of the arch.

Also find the horizontal and the vertical components of the reaction at the joints B and C.

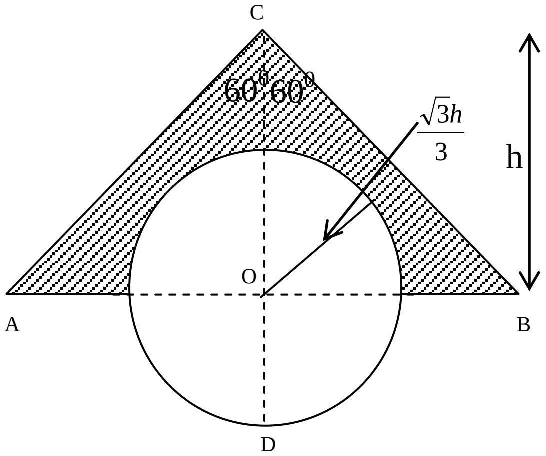
(b)



The diagram represents a framework of seven smoothly jointed light rods hinged to a fixed point at A. The outer rods form a square and A, C, B are collinear with AC = 3CB . Draw a stress diagram using Bow's notation and determine the stresses in the rods BC and AE in terms of P, indicating whether each stress is a tension or a thrust.

(7) Find by using integration that the centre of mass of a uniform solid right circular cone of weight W, height h and semi-vertex angle 60^0 from the centre of the circular base O.

From the above cone, a hemisphere of radius $\frac{\sqrt{3}h}{3}$ is removed from the base of the above cone so that the centre of the hemisphere coincide with the centre of the circular base O. Showthat the centre of mass of the remaining part lies at a distance from O is $\frac{6h}{27-2\sqrt{3}}$.



The body shown in the figure consists of the remaining part and the hemisphere of centre O and radius $\frac{\sqrt{3}h}{3}$ made of the same material are rigidly joined to form a composite body as shown in the figure which is suspended from A.

Find in the position of equilibrium the base of the cone makes an angle with the vertical. The weight W_0 which if suspended from D will make AB vertical then showthat $3\sqrt{3}W_0 = 2W$.

(8)(a) Define the conditional probability *P(A/B)* of an event *A*, given *B*.

Showthat i) $P(A' / B) = 1 - P(A/B)$ ii) $P(A/B') = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

iii) $P\left(A/_B\right)P(B) + P\left(A/_{B'}\right)P(B') = P(A)$

Where A' and B' are complimentary events of A and B.

(b) If *A* and *B* are mutually exclusive events then showthat,

$$P(A/A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

(c) If *A* and *B* are two independents events, with probabilities 0.5 and 0.2 respectively.
Find, i) $P(A/A \cup B)$ ii) $P((A \cap B)/A' \cup B')$

(d) There players A, B and C alternately throw a die in that order, the first player to throw a 6 being deemed the winner. A's die is fair, where as B and C throw biased dice with probabilities a and b respectively, of throwing a 6. Find the probability that A wins the game. What should be the values of a and b so that the game is equiprobable for all three players.

(9)(a) Let $\{x_1, x_2, ..., x_n\}$ be a set of *n* observations.

Define the mean $\bar{x}$ and standard deviation 6 of the set.

The standard deviation of the set of 100 observations $x_1, x_2, ..., x_{100}$ is 8. Determine the standard deviation of the set $\{2x_1 + 3, 2x_2 + 3, ..., 2x_{100} + 3\}$.

(b) The speed of 100 buses entering Colombo city were measured in km/h and the results are given in the following table.

Mid value of the class interval x_i	25	30	35	40	45	50	55	60
Frequency f_i	5	11	15	16	17	13	12	11

i) Write down the class boundaries and class limits of the 4th class.

ii) Find the modal class and hence the mode of the distribution.

iii) Calculate the mean and standard deviation of the distribution.

iv) Find the coefficient of skewness of the distribution. What is the shape of this distribution.