G.C.E.(A.L) Support Seminar - 2015

Combined Mathematics I

Answer Guide

Part A

1. When $n = 1$, L.H.S $= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

R.H.S $= \begin{pmatrix} 1+2\times 1 & -4\times 1 \\ 1 & 1-2\times 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} =$ L.H.S

$\therefore$ The result is true for $n = 1$. 5

Assume that the result is true for $n = p$.

Then, $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{p} = \begin{pmatrix} 1+2p & -4p \\ p & 1-2p \end{pmatrix}$ 5

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{p+1} = \begin{pmatrix} 1+2p & -4p \\ p & 1-2p \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \quad 5$$

$$= \begin{pmatrix} 3(1+2p)-4p & -4(1+2p)+4p \\ 3p+1-2p & -4p-(1-2p) \end{pmatrix}$$

$$= \begin{pmatrix} 3+2p & -4-4p \\ 1+p & -1-2p \end{pmatrix}$$

$$= \begin{pmatrix} 1+2(p+1) & -4(p+1) \\ (p+1) & 1-2(p+1) \end{pmatrix} \quad 5$$

$\therefore$ The result is true for $n = p + 1$.

$\therefore$ By the Principle of Mathematical Induction, the result is true for all positive integers n. 5

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2. $\left(\sqrt{5}+\sqrt{3}\right)^4 + \left(\sqrt{5}-\sqrt{3}\right)^4$

$$= {}^4C_0\left(\sqrt{5}\right)^4 + {}^4C_1\left(\sqrt{5}\right)^3.\sqrt{3} + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{3}\right)^4 + {}^4C_0\left(\sqrt{5}\right)^4 \quad 5$$
$$- {}^4C_1\left(\sqrt{5}\right)^3\left(\sqrt{3}\right) + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 - {}^4C_3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{3}\right)^4$$

$$= 2\left[{}^4C_0\left(\sqrt{5}\right)^4 + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_4\left(\sqrt{3}\right)^4\right]$$

$= 2[25 + 6\times 15 + 9]$ 5

$= 2\times 124$

$= 248$ ——— (1)

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$$\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right) = 5-3=2 \quad 5$$

$$\Rightarrow \left(\sqrt{5}-\sqrt{3}\right) = \frac{2}{\sqrt{5}+\sqrt{3}}$$

$$\Rightarrow 0<\left(\sqrt{5}-\sqrt{3}\right)<1 \quad 5 \quad \left(\because \sqrt{5}+\sqrt{3}>2\right)$$

$$\Rightarrow 0<\left(\sqrt{5}-\sqrt{3}\right)^4<1$$

$\therefore$ From (1), $0<248-\left(\sqrt{5}+\sqrt{3}\right)^4<1$

$$\Rightarrow 247<\left(\sqrt{5}+\sqrt{3}\right)^4<248$$

$$\Rightarrow n=247 \quad 5$$

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3. $(3-2i)(7-5i) = 21+10i^2-14i-15i$ 5

$= 11-29i \quad (\text{since } i^2=-1)$ 5

$\therefore 11+29i = (3+2i)(7+5i)$ 5

$11^2+29^2 = 11^2-(29i)^2$

$= (11-29i)(11+29i)$

$= (3-2i)(7-5i)(3+2i)(7+5i)$

$= (9-4i^2)(49-25i^2)$ 5

$= (9+4)(49+25)$

$= 13\times 74$ 5

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4. $$\lim_{x\to\frac{\pi}{2}} \frac{(2x-\pi)\cos x}{2\cos^2 x-\left(\frac{\pi}{2}-x\right)^2\sin x}$$

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{2\left(x-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}-x\right)}{2\sin^2\left(\frac{\pi}{2}-x\right)-\left(\frac{\pi}{2}-x\right)^2\cos\left(\frac{\pi}{2}-x\right)}$$

5

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{-2\sin\left(x-\frac{\pi}{2}\right)\Big/\left(x-\frac{\pi}{2}\right)}{\dfrac{2\sin^2\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)^2}-\cos\left(x-\frac{\pi}{2}\right)} \quad 5$$

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{-2\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}}{2\left[\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right]^2 - \cos\left(x-\frac{\pi}{2}\right)}$$

$$= \frac{-2\left[\lim\limits_{\left(x-\frac{\pi}{2}\right)\to 0} \dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right] \quad 5}{2\left[\lim\limits_{\left(x-\frac{\pi}{2}\right)\to 0} \dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right]^2 - \lim\limits_{\left(x-\frac{\pi}{2}\right)\to 0} \cos\left(x-\frac{\pi}{2}\right) \quad 5}$$

$$= \frac{-2\times 1}{2(1)^2-1} \quad 5$$

$$= -2$$

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5. $$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x+\sqrt{x^2+a^2}\right) = \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\cdot\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2+a^2}}\right) \quad 5$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\cdot\left(\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{\sqrt{x^2+a^2}} \quad 5$$

Then $$\mathrm{I} = \int\frac{1}{\sqrt{9x^2+4}}\,\mathrm{d}x = \frac{1}{3}\int\frac{1}{\sqrt{x^2+\left(\frac{2}{3}\right)^2}}\,\mathrm{d}x \quad 5$$

$$\therefore \mathrm{I} = \frac{1}{3}\left[\ln\left(x+\sqrt{x^2+\left(\frac{2}{3}\right)^2}\right)\right]+C$$; Here C is an arbitrary constant.

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6. $\left(\frac{dy}{dx}\right)_T \quad = \quad \frac{T(2-3T)}{(1-T)(1-3T)}$

When $T = \frac{1}{2}$

$\left(\frac{dy}{dx}\right) \quad = \quad \frac{\frac{1}{2}\left(2-\frac{3}{2}\right)}{\left(1-\frac{1}{2}\right)\left(1-\frac{3}{2}\right)} \quad = \quad -1$ 5

Then $(x, y) \quad = \quad \left(\frac{1}{8}, \frac{1}{8}\right)$ 5

If any point on the tangent is given by (x, y), the equation of the tangent is

$y-\frac{1}{8} \quad = \quad -1\left(x-\frac{1}{8}\right)$ 5

$4x + 4y - 1 = 0$

Let the tangent and the curve intersect at the point corresponding to $t = T'$.

Then $(x\,, y) \quad = \quad \left(T'(1-T')^2, T'^2(1-T')\right)$

$\Rightarrow \qquad 4T'(1-T')^2 + 4T'^2(1-T') - 1 = 0$ 5

$4T'^2 - 4T' + 1 = 0$

$(2T'-1)^2 = 0$

$\Rightarrow \; T' = \frac{1}{2}$; This is the parameter corresponding to the given point. 5

Therefore the tangent does not meet the curve again.

That is, the curve is located on one side of the tangent $4x + 4y - 1 = 0$.

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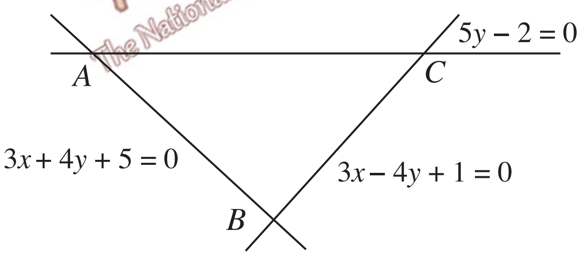
7.



The bisectors of $A\hat{B}C$ are given by $\left|\frac{3x+4y+5}{\sqrt{3^2+4^2}}\right| = \left|\frac{3x-4y+1}{\sqrt{3^2+(-4)^2}}\right|$. 10

i.e., $\quad 3x + 4y + 5 \quad = \quad \pm(3x - 4y + 1)$

$8y + 4 = 0$ and $6x + 6 = 0$

$2y + 1 = 0$ and $x + 1 = 0$ 5

Since the straight line given by $2y + 1 = 0$ is parallel to the side AC, it is the exterior bisector. 5

The straight line $x + 1 = 0$ is perpendicular to the side AC. It is the interior bisector of $A\hat{B}C$. 5

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8. For $ax^2 + 2y^2 + bxy + x + 4y + 2c = 0$ to represent a circle,

$a = 2, b = 0$ and $\left(\frac{1}{4}\right)^2 + 1^2 - c > 0$. 5

$\therefore\ a = 2, b = 0$ and $c < \frac{17}{16}$. 5

$\therefore$ The positive integral value of c is 1.

Then the circle is $2x^2 + 2y^2 + x + 4y + 2 = 0$

That is, $x^2 + y^2 + \frac{1}{2}x + 2y + 1 = 0$

The centre is $\left(-\frac{1}{4}, -1\right)$ 5

$(x+p)^2 + y^2 = p^2 \Rightarrow x^2 + y^2 + 2px = 0$

The common chord of the two circles is $(\frac{1}{2} - 2p)x + 2y + 1 = 0$

Since the given circle is bisected, its centre must be located on the common chord. 5

$\therefore \left(\frac{1}{2} - 2p\right)\left(-\frac{1}{4}\right) + 2(-1) + 1 = 0$

$\frac{1}{2}p - \frac{1}{8} - 2 + 1 = 0$

$p = \frac{9}{4}$ 5

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9. $S_1 : x^2 + y^2 - 6x + 8y + 9 = 0 \Rightarrow C_1 = (3, -4)$, $r_1 = \sqrt{9+16-9} = 4$

$S_2 : x^2 + y^2 - r^2 = 0 \Rightarrow C_2 = (0, 0)$, $r_2 = r$ 5

For the two circles S_1 and S_2 to touch each other $C_1C_2 = r_1 + r_2$ or $C_1C_2 = |r_1 - r_2|$

$5 = 4 + r$ or $5 = |r - 4|$

$r = 1$ 5 $\quad r - 4 = \pm 5$

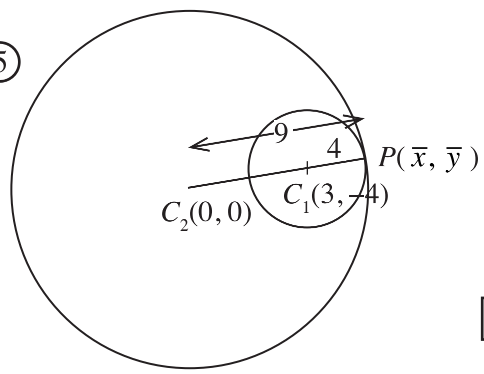
$r = 4 \pm 5$

Since $r > 0, r = 9$ 5

When the circles S_1 and S_2 touch each other internally, the straight line $C_2\,C_1$ is divided externally at the point of contact P in the ratio $C_2P : PC_1 = 9 : 4$ 5

$\therefore P \equiv \left[\frac{9 \times 3 - 4 \times 0}{9 - 4}, \frac{9 \times (-4) - 4 \times 0}{9 - 4}\right]$

$\equiv \left(\frac{27}{5}, -\frac{36}{5}\right)$ 5



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10. The cosine formula gives,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$16 = 25 + 36 - 60\cos C \quad 5$$

$$\cos C = \frac{45}{60}$$

$$= \frac{3}{4}$$

The sine formula gives,

$$\frac{6}{\sin B} = \frac{4}{\sin C} \quad 5$$

$$\Rightarrow \sin B = \frac{3}{2}\sin C \qquad \therefore \frac{\sin B}{\sin C} = \frac{3}{2}.$$

$$= 2 \times \frac{3}{4} \times \sin C$$

$$= 2\cos C \sin C \quad 5$$

$$= \sin 2C$$

$\therefore\ \hat{B} = 2\hat{C}$ or $\hat{B} = \pi - 2\hat{C}$; (since $0 < \hat{A}, \hat{B}, \hat{C} < \pi$) 5

Since $\hat{A} \neq \hat{C}$, $\hat{B} \neq \pi - 2\hat{C}$; (since $\hat{A} + \hat{B} + \hat{C} = \pi$) 5

$\therefore\ \hat{B} = 2\hat{C}$

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Part B

11. (a) $\alpha+\beta=-b,\ \ \alpha\beta=c$ 5

$p+q = \alpha+\beta+\alpha^2+\beta^2 = (\alpha+\beta)+(\alpha+\beta)^2-2\alpha\beta = b^2-b-2c$ 5

$pq = \alpha^3+\beta^3+\alpha\beta+(\alpha\beta)^2 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)+\alpha\beta+(\alpha\beta)^2$

$= -b^3+3cb+c+c^2$ 5

$\therefore$ The quadratic equation whose roots are p and q is,

$x^2-(b^2-b-2c)x-b^3+3cb+c+c^2 = 0$ 5 + 5

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The discriminant,

$\Delta_x = (b^2-b-2c)^2+4(b^3-3bc-c-c^2)$ 5

$= b^4+b^2+4c^2-2b^3+4bc-4b^2c+4b^3-12bc-4c-4c^2$

$= b^4+2b^3+b^2-4b^2c-8bc-4c$

$= b^2(b^2+2b+1)-4c(b^2+2b+1)$

$= (b+1)^2(b^2-4c)$ 5

When α and β are imaginary, $b^2-4c<0$. 5

$\therefore\ \Delta_x \le 0$ 5

$\therefore\ \Delta_x = 0$ if and only if $b = -1$. 5

That is, p and q are real, if and only if $b = -1$.

Since $\Delta_x = 0$, $p = q$. 5

Then $p+q = (-1)^2-(-1)-2c$ 5

$2p = 2-2c$

$p = 1-c$

$\therefore\ p = q = 1-c$

$\boxed{35}$

(b) Let $y=\dfrac{(x+2)^2}{x^2+x+1}$ 5

$\Rightarrow\quad (y-1)x^2+(y-4)x+(y-4) = 0$ 5

If $y=1$, then $x=-1$. In this case a quadratic equation does not exist.

$$\Rightarrow\quad (y-1)\left[\left(x+\frac{y-4}{2(y-1)}\right)^2+\frac{(y-4)}{(y-1)}-\left(\frac{y-4}{2(y-1)}\right)^2\right]=0 \qquad (y\neq 1)$$

$$\Rightarrow\quad (y-1)\left[\left(x+\frac{y-4}{2(y-1)}\right)^2+\frac{4(y-1)(y-4)-(y-4)^2}{4(y-1)^2}\right]=0$$

$$\Rightarrow\quad (y-1)\left[\left(x+\frac{y-4}{2(y-1)}\right)^2+\frac{(y-4)(4y-4-y+4)}{4(y-1)^2}\right]=0$$

$$\Rightarrow\quad (y-1)\left[\left(x+\frac{y-4}{2(y-1)}\right)^2+\frac{3y(y-4)}{4(y-1)^2}\right]=0$$ 5

For all real x, $\dfrac{3y(y-4)}{4(y-1)^2} \le 0$

$\Rightarrow \quad 3y(y-4) \le 0$, (5) since $(y-1)^2 > 0$

$\Rightarrow \quad 0 \le y \le 4$ (5)

$\Rightarrow \quad 0 \le \dfrac{(x+2)^2}{x^2+x+1} \le 4$

$\Rightarrow \quad y_{\min} = 0$ and $y_{\max} = 4$ (5)

The graph of $y = \dfrac{(x+2)^2}{x^2+x+1}$ is continuous, since $x^2+x+1 = \left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$.

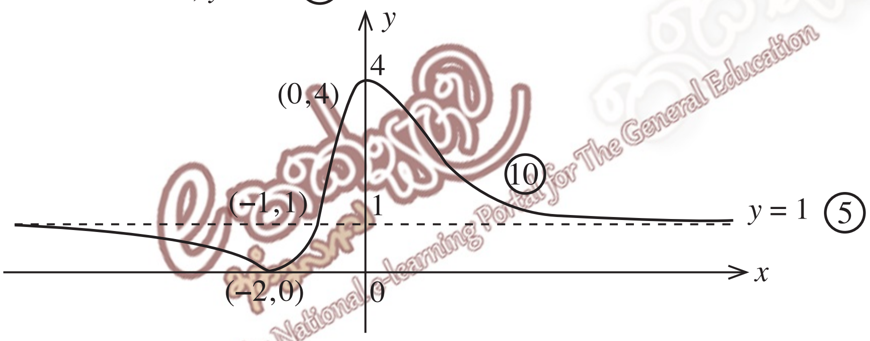
$(0, 4)$ is a maximum. (5)

$(-2, 0)$ is a minimum. (5)

When $y = 1$, $x^2 + x + 1 = x^2 + 4x + 4 \Rightarrow 3x + 3 = 0 \Rightarrow x = -1$

$$y = \frac{x^2+4x+4}{x^2+x+1} = \frac{1+\frac{4}{x}+\frac{4}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}}, \quad \text{for } x \ne 0$$

When $x \to \pm\infty$, $y \to 1$ (5)



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(c) If $x^2 + kx + 1$ is a factor of $x^4 - 12x^2 + 8x + 3$, there is $\lambda \in \mathbb{Z}$ such that,

$x^4 - 12x^2 + 8x + 3 \equiv (x^2 + kx + 1)(x^2 + \lambda x + 3)$

(5)

$\Rightarrow \quad k + \lambda = 0$ and $\lambda + 3k = 8$ (5)

$\Rightarrow \quad k = 4$ (5)

$\Rightarrow \quad \lambda = -4$

$x^4 - 12x^2 + 8x + 3 = 0$

$(x^2 + 4x + 1)(x^2 - 4x + 3) = 0$ (5)

$x^2 + 4x + 1 = 0$ or $x^2 - 4x + 3 = 0$

$x = \dfrac{-4 \pm \sqrt{16-4}}{2}$ or $(x-1)(x-3) = 0$

$= -2 \pm \sqrt{3}$ (5) or $x = 1$ or $x = 3$ (5)

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12. (a) Since each child should get at least three rupees, an amount of 15 rupees must be reserved compulsorily. Then the remaining three rupees can be divided among the five children in different ways as given below. 5

	Division of money						No. of ways	
(i)	3	0	0	0	0	5	$\frac{5!}{4!} = 5$	5
(ii)	2	1	0	0	0	5	$\frac{5!}{3!} = 20$	5
(iii)	1	1	1	0	0	5	$\frac{5!}{3!2!} = 10$	5

$\therefore$ Total number of ways $= 5 + 20 + 10$ 5
$= 35$

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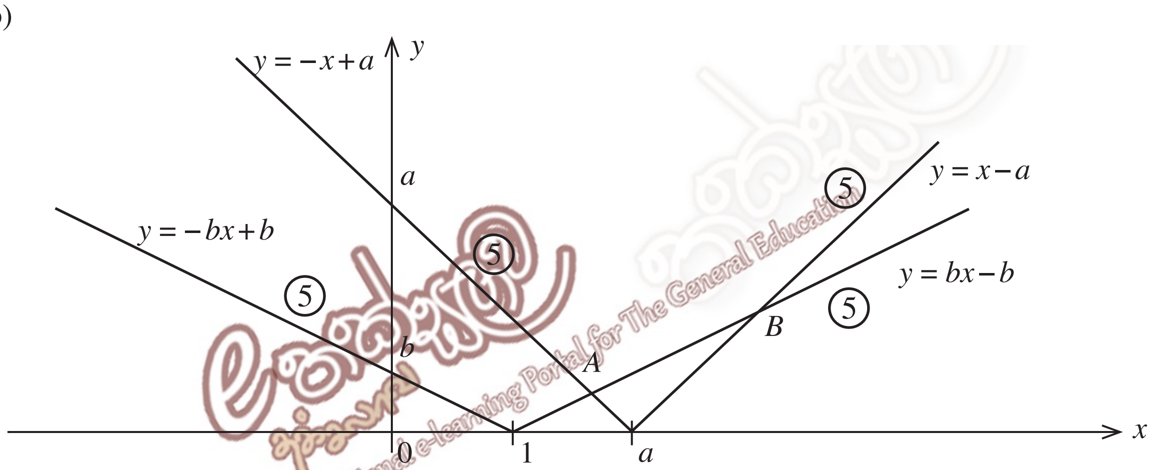
(b)



$a > b > 0$

Since the solution set of $b|x-1| > |x-a|$ is $\{x \mid 3 < x < 7 \,;\, x \in \mathbb{R}\}$,

$A \equiv (3, y_A)$; $B \equiv (7, y_B)$. 5

By considering y_A, $-3 + a = 3b - b \;\Rightarrow\; a - 2b = 3$ ——— (1) 5

Similarly by considering y_B, $7 - a = 7b - b \Rightarrow\; a + 6b = 7$ ——— (2) 5

From (1) and (2), $a = 4,\; b = \frac{1}{2}$ 5

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(c) Let $u_r \equiv \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3} \equiv \frac{3r+1}{(r+1)(r+2)(r+3)}$. 5

Then $A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2) \equiv 3r+1$

By considering the coefficients of,

$$\left.\begin{array}{ll} r^2\ ; & A+B+C=0 \\ r\ ; & 5A+4B+3C=3 \\ \text{Constant} & 6A+3B+2C=1 \end{array}\right\} 10$$

By solving the above equations ; $A = -1, B = 5, C = -4$ 10

$\therefore u_r \equiv -\frac{1}{r+1} + \frac{5}{r+2} - \frac{4}{r+3}$

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$$\equiv -\left(\frac{1}{r+1}-\frac{1}{r+2}\right)+4\left(\frac{1}{r+2}-\frac{1}{r+3}\right) \quad (5)$$

This is in the form of $\lambda[f(r)-f(r+1)] + \mu[f(r+1)-f(r+2)]$.

Here $\lambda = -1$, $\mu = 4$ and $f(r) = \left(\frac{1}{r+1}\right)$. (5)
(5)

$$\begin{aligned}
u_r &\equiv \lambda[f(r)-f(r+1)] + \mu[f(r+1)-f(r+2)] \\
\therefore u_1 &= \lambda[f(1)-f(2)] + \mu[f(2)-f(3)] \\
u_2 &= \lambda[f(2)-f(3)] + \mu[f(3)-f(4)] \quad (5) \\
&\vdots \\
u_{n-1} &= \lambda[f(n-1)-f(n)] + \mu[f(n)-f(n+1)] \\
u_n &= \lambda[f(n)-f(n+1)] + \mu[f(n+1)-f(n+2)] \quad (5)
\end{aligned}$$

By adding,

$$\begin{aligned}
\sum_{r=1}^{n} u_r &= \lambda[f(1)-f(n+1)] + \mu[f(2)-f(n+2)] \\
&= -1\left[\frac{1}{2}-\frac{1}{n+2}\right]+4\left[\frac{1}{3}-\frac{1}{n+3}\right] \quad (5) \\
&= \frac{5}{6}+\frac{1}{n+2}-\frac{4}{n+3} \\
&= \frac{5}{6}-\frac{3n+5}{(n+2)(n+3)}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{r=1}^{n} u_r &= \lim_{n\to\infty}\left[\frac{5}{6}+\frac{1}{n+2}-\frac{4}{n+3}\right] \\
&= \frac{5}{6} \quad (5)
\end{aligned}$$

$\Rightarrow$ Since $\lim_{n\to\infty}\sum_{r=1}^{n} u_r$ is finite, the series is convergent. (5)

Further more, for all $r \in \mathbb{Z}^+$, $u_r > 0$.

$\therefore u_1 \leq S_n < S_\infty$ (5)

$$\Rightarrow \frac{1}{6} \leq \frac{5}{6}-\frac{3n+5}{(n+2)(n+3)} < \frac{5}{6}$$

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13. (a) $\mathbf{P} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$

$$\mathbf{P} - \lambda\mathbf{I} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{pmatrix} \qquad 5$$

$$\det(\mathbf{P} - \lambda\mathbf{I}) = 0$$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = 0 \qquad 5$$

$$\Rightarrow (5+\lambda)(2+\lambda) - 18 = 0 \qquad 5$$

$$\Rightarrow \lambda^2 + 7\lambda - 8 = 0$$

$$\Rightarrow (\lambda-1)(\lambda+8) = 0$$

$$\Rightarrow \lambda = 1 \quad \text{or} \quad \lambda = -8 \qquad 5$$

$\mathbf{PX} = \lambda\mathbf{X}$

$$\Rightarrow \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \begin{pmatrix} -5x+3y \\ 6x-2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

When $\lambda = 1$, $\left.\begin{aligned} -5x + 3y &= x \\ 6x - 2y &= y \end{aligned}\right\} \Rightarrow \begin{aligned} -6x + 3y &= 0 \\ 6x - 3y &= 0 \end{aligned}$ 5

Since the two equations above are equivalent,
when $x = t$, $y = 2t$; Here t is a real parameter.

$$\therefore X = \begin{pmatrix} t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad 5$$

When $\lambda = -8$, $\left.\begin{aligned} -5x + 3y &= -8x \\ 6x - 2y &= -8y \end{aligned}\right\} \Rightarrow \begin{aligned} 3x + 3y &= 0 \\ 6x + 6y &= 0 \end{aligned}$ 5

Since the above two equations are also equivalent,
when $x = T$ then $y = -T$; Here T is a real parameter.

$$\therefore X = \begin{pmatrix} T \\ -T \end{pmatrix} = T\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad 5$$

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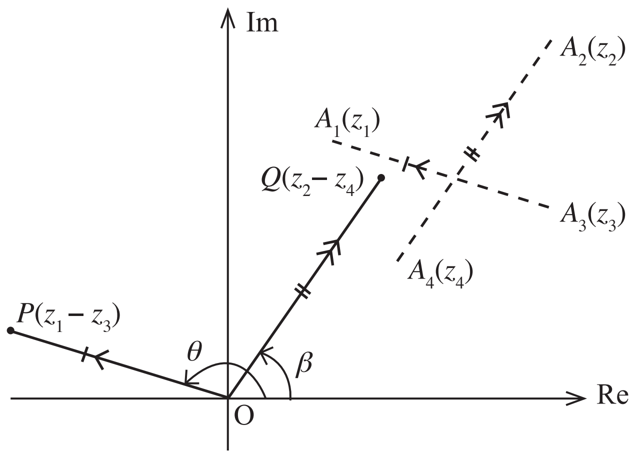
(b)



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P and Q are two points such that $OP \parallel A_3A_1$, $OP = A_3A_1$ and $OQ \parallel A_4A_2$, $OQ = A_4A_2$ respectively.

Then $z_P = z_1 - z_3$ and $z_Q = z_2 - z_4$. 10

$$\left|\frac{z_1 - z_3}{z_2 - z_4}\right| = \frac{|z_1 - z_3|}{|z_2 - z_4|} = \frac{OP}{OQ} = \frac{A_1A_3}{A_2A_4} \quad 10$$

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_4}\right) = \arg(z_1 - z_3) - \arg(z_2 - z_4) = \theta - \beta = P\hat{O}Q$$

When $\beta > \theta$, $|\theta - \beta| = P\hat{O}Q$.

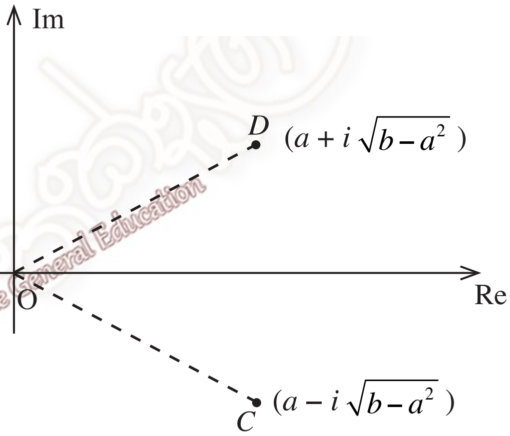
$P\hat{O}Q$ = the angle between A_1A_3 and A_2A_4. 10

Therefore, for $\frac{z_1 - z_3}{z_2 - z_4}$ to be purely imaginary, $A_1A_3 \perp A_2A_4$. 5

$\boxed{35}$

$z^2 - 2z + 2 = 0 \quad \Rightarrow \quad z = 1 \pm i$

$z^2 - 2az + b = 0 \quad \Rightarrow \quad z = a \pm i\sqrt{b - a^2}$



(i) If $C\hat{O}D = \frac{\pi}{2}$,

$\frac{a + i\sqrt{b-a^2}}{a - i\sqrt{b-a^2}}$ is purely imaginary. 5

$$\frac{a + i\sqrt{b-a^2}}{a - i\sqrt{b-a^2}} = \frac{a^2 - (b - a^2) + 2ia\sqrt{b-a^2}}{a^2 + (b - a^2)}$$

For this number to be purely imaginary, the real part $= 0 \Rightarrow 2a^2 = b$ 10

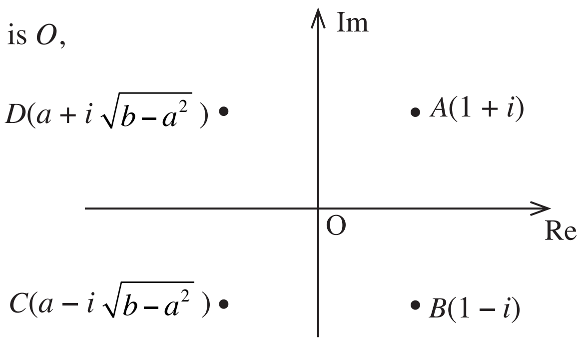
(ii) $OA = OB = OC = OD$

$\therefore\ z_A\bar{z}_A = z_D\bar{z}_D$ 5

$\Rightarrow\ 2 = a^2 + b - a^2$ 10

$\Rightarrow\ b = 2$

(iii) When $ABCD$ is a square and its centre is O,
by symmetry, 5
$a = -1$ and $\sqrt{b-a^2} = 1$
$\Rightarrow\ b = 2$ 10



$\boxed{45}$

[see page thirteen

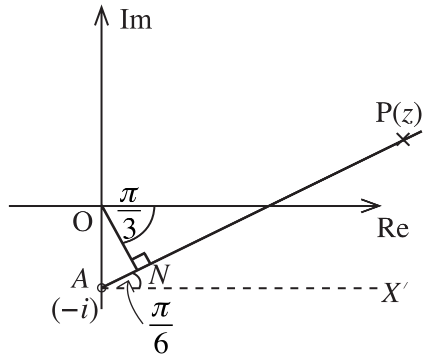
(c) $\arg[(z+i)i]=\frac{2\pi}{3}$

$\arg(z+i)+\arg i = \frac{2\pi}{3}$

$\therefore \arg(z+i) = \frac{2\pi}{3}-\frac{\pi}{2} = \frac{\pi}{6}$ (5)

$\Rightarrow \arg(z-(-i)) = \frac{\pi}{6}$ (5)

$\therefore P\hat{A}X' = \frac{\pi}{6}\,;\, P \neq A$ (5)



The locus of P is a line segment as shown in the figure.

$OP = |z|$ (5)

$\therefore |z|_{\text{least}} = ON = 1.\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ (5)

$z_N = \left\{\frac{\sqrt{3}}{2}\cos\frac{\pi}{3} - i\frac{\sqrt{3}}{2}\sin\frac{\pi}{3}\right\}$

$= \frac{\sqrt{3}}{4}-\frac{3}{4}i$ (5) $\boxed{30}$

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14. (a) $f(x)=\frac{3-4x}{x^2+1}\,,\; x\in\mathbb{R}$

$f'(x)=\frac{(x^2+1)(-4)-(3-4x)2x}{(x^2+1)^2}$ (10) $=\frac{2(2x+1)(x-2)}{(x^2+1)^2}$

Since at $x=-\frac{1}{2}$ and $x=2$, $f'(x)=0$, (10) there exist two turning points.

For all $x\in\mathbb{R}$, the function f is continuous.

x	$-\infty<x<-\frac{1}{2}$	$-\frac{1}{2}<x<2$	$2<x<\infty$
$f'(x)$	$\frac{(-)(-)}{(+)}>0$	$\frac{(+)(-)}{(+)}<0$	$\frac{(+)(+)}{(+)}>0$

(10)

$x=-\frac{1}{2}\,,\; y=4$ is a maximum. (5)

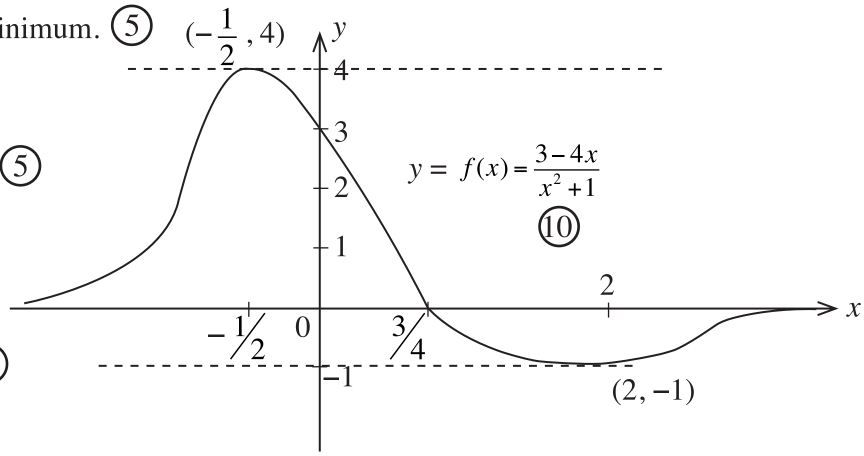
$x=2,\, y=-1$ is a minimum. (5)

When $x=0,\; y=3$

When $x=\frac{3}{4}\,,\, y=0$ (5)

$y=\dfrac{\frac{3}{x^2}-\frac{4}{x}}{1+\frac{1}{x^2}}$

$x\to\pm\infty,\; y\to 0$ (5)



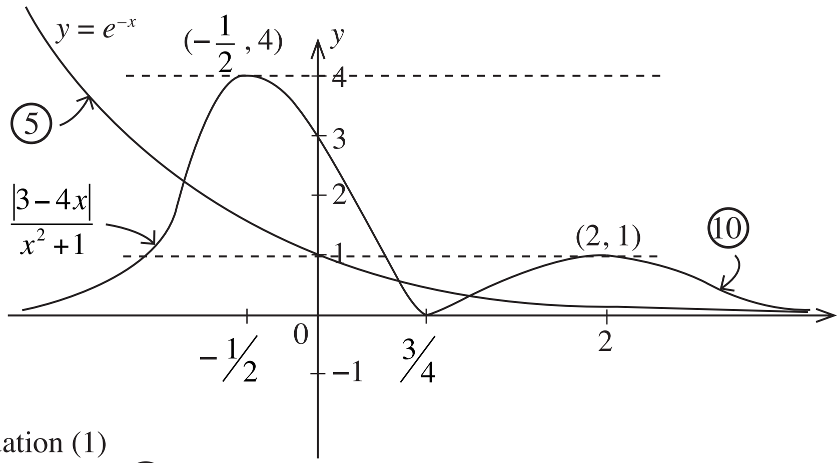
$|3-4x|e^x - x^2 - 1 = 0$ ——— (1)

[see page fourteen

$$e^{-x} = \frac{|3-4x|}{x^2+1} \quad (5)$$

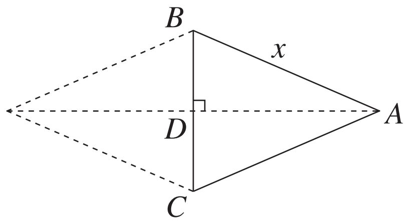
The roots of the equation (1) are given by the x - coordinates of the intersection points of the curves $y = e^{-x}$ and $y = \dfrac{|3-4x|}{x^2+1}$. (5)

According to the graph, equation (1) has at least three distinct real roots. (5)



$\boxed{90}$

(b)



Let the volume of the generated solid be V, when $AB = x$.

Then $BC = 2s - 2x$,

$$BD = \frac{2s-2x}{2} = s - x \quad (5)$$

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{x^2 - (s-x)^2} = \sqrt{2sx - s^2}$$

$$= \sqrt{s(2x-s)} \quad (5) \; ; \; x > \frac{s}{2} \quad (5)$$

$$V = \frac{1}{3}\pi AD^2 . 2BD \quad (5)$$

$$= \frac{1}{3}\pi s(2x-s).2(s-x) \quad (5)$$

$$= \frac{2\pi}{3} s(2x-s)(s-x)$$

$$\frac{dV}{dx} = \frac{2}{3}\pi s[(2x-s)(-1)+(s-x)2] \quad (5)$$

$$= \frac{2}{3}\pi s[-2x+s+2s-2x]$$

$$= \frac{2}{3}\pi s(3s-4x) \quad (5)$$

When $x = \frac{3}{4}s$, $\frac{dV}{dx} = 0$. (5)

When $\frac{s}{2} < x < \frac{3s}{4}$, $\frac{dV}{dx} > 0$; $\therefore V$ increases.

When $\frac{3s}{4} < x < s$, $\frac{dV}{dx} < 0$; $\therefore V$ decreases. (10)

$\therefore$ When $x = \frac{3}{4}s$, V is a maximum. (5)

$\therefore$ The volume is maximum, when AB is of length $\frac{3}{4}s$. (5)

$\boxed{60}$

15. (a) $\dfrac{x^2+3x+5}{(x-1)(x+2)} \equiv A+\dfrac{B}{x-1}+\dfrac{C}{x+2}$

$\Rightarrow x^2+3x+5 \equiv A(x-1)(x+2)+B(x+2)+C(x-1)$

By comparing the coefficents of x^2, $A=1$ 5

When $x=1$, $B=3$ 5

When $x=-2$, $C=-1$ 5

$$\therefore \int_0^2 \frac{x^2+3x+5}{(x-1)(x+2)}\,dx = \int_0^2 dx + 3\int_0^2 \frac{1}{x-1}\,dx - \int_0^2 \frac{1}{x+2}\,dx \quad 5$$

$$= [x]_0^2 + 3\Big[\ln|x-1|\Big]_0^2 - \Big[\ln|x+2|\Big]_0^2 \quad 10$$

$$= 2 + 3\,[\ln 1 - \ln 1] - [\ln 4 - \ln 2]$$

$$= 2 - \ln 2 \quad 5$$

$\boxed{35}$

(b) $$\int e^{2x}\sin 3x\;dx = \int \sin 3x\,\frac{d}{dx}\left(\frac{e^{2x}}{2}\right)dx \quad 5$$

$$= \frac{1}{2}e^{2x}\sin 3x - \int \frac{1}{2}e^{2x}\,3\cos 3x\;dx + C \quad 10$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x - \frac{1}{2}\int e^{2x}(-3\sin 3x)\,dx\right] + C' \quad 10$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\;dx + C' \quad 5$$

$$\Rightarrow \frac{13}{4}\int e^{2x}\sin 3x\;dx = \frac{1}{4}e^{2x}(2\sin 3x - 3\cos 3x) + C' \quad 10$$

$\Rightarrow \displaystyle\int e^{2x}\sin 3x\;dx = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$; Here C is an arbitrary constant.

5

$\boxed{45}$

(c) $$I = \int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,dx$$

$$x = \sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

When $x=1$, $\theta=\dfrac{\pi}{2}$

When $x=0$, $\theta=0$ 5

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\sqrt{1-\sin^2\theta}}\,d\theta \quad 5$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta \;\text{———}\; (1)$$

[see page sixteen

When $\theta = \dfrac{\pi}{2} - t,$

$\dfrac{d\theta}{dt} = -1$ 5

When $\theta = 0$, $t = \dfrac{\pi}{2}$

When $\theta = \dfrac{\pi}{2}$, $t = 0$ 5

$$\therefore I = \int_{\frac{\pi}{2}}^{0} \frac{\sin t}{\cos t + \sin t}(-dt)$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sin t}{\cos t + \sin t}\, dt$$ 5

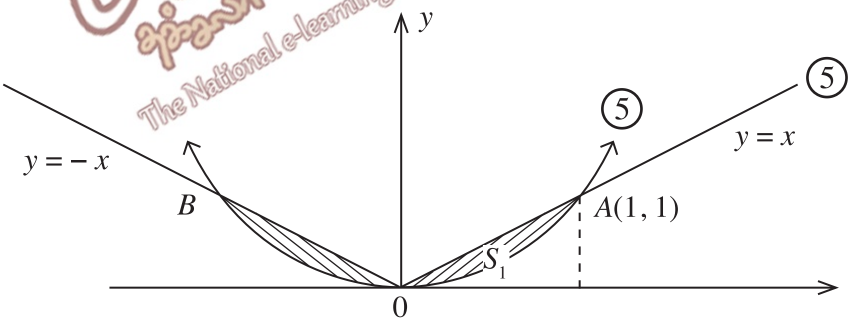
Replacing t by θ, $I = \displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta}\, d\theta$ ——— (2) 5

(1) + (2) ; $2I = \displaystyle\int_{0}^{\frac{\pi}{2}} d\theta = [\theta]_0^{\pi/2}$ 5 + 5

$\therefore I = \dfrac{\pi}{4}$

$\boxed{40}$

(d) Let $S = \{(x, y) : x^2 \le y \le |x|\}$.



By solving $y = x$ and $y = x^2$,

$A \equiv (1, 1)$ 5

$S_1 = \dfrac{1}{2} \times 1 \times 1 - \displaystyle\int_0^1 x^2\, dx$ 5

$= \dfrac{1}{2} - \left[\dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{2} - \left(\dfrac{1}{3}\right)$ 5

$= \dfrac{3-2}{6} = \dfrac{1}{6}$

$\therefore$ The enclosed area $= 2S_1 = \dfrac{1}{3}$ square units 5

$\boxed{30}$

[see page seventeen

16. (a) $A \equiv \left(\frac{4}{3}, \frac{5}{3}\right)$ (5)

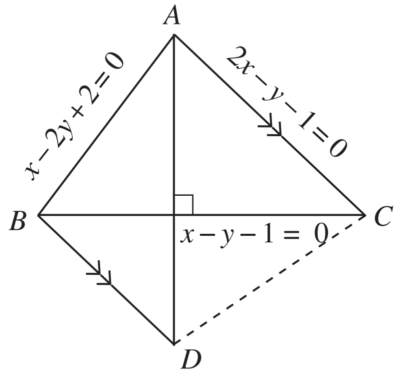
$B \equiv (4, 3)$ (5)

The equation of the line AD,

$$y-\frac{5}{3} = -1\left(x-\frac{4}{3}\right) \quad (10)$$

$$y-\frac{5}{3} = -x+\frac{4}{3}$$

$$x+y-3 = 0 \quad (5)$$



Since $BD \,//\, AC$, the equation of the line BD can be taken as $2x-y-k = 0$. Here k is a constant. (5)

Since this passes through B, $8-3-k = 0$

$\therefore\ k = 5$ (5)

$\therefore$ the equation of the line BD is $2x-y-5 = 0$ (5)

$C \equiv (0, -1)$ (5)

$D \equiv \left(\frac{8}{3}, \frac{1}{3}\right)$ (5)

$$\therefore \text{ the gradient of } CD = \frac{\frac{1}{3}+1}{\frac{8}{3}-0} = \frac{1}{2} \quad (5)$$

$$\text{The gradient of } AB = \frac{3-\frac{5}{3}}{4-\frac{4}{3}} = \frac{1}{2} \quad (5)$$

$\therefore AB \,//\, CD$ (5)

Also, since $BD \,//\, AC$, $ABCD$ is a parallelogram. (5)

Since its diagonals are perpendicular, $ABDC$ is a rhombus. (5)

$\boxed{75}$

(b) $|r_1-r_2| < C_1C_2 < r_1+r_2$ (10)

If the circles $x^2+y^2+6x+2fy=0$ and $x^2+y^2-2y-3=0$ intersect each other orthogonally,

$2.3.0+2f(-1) = 0+(-3)$ (10)

$\Rightarrow\ f = \frac{3}{2}$

Let the circles $S_1 \equiv x^2+y^2+6x+2fy=0$ and $S_2 \equiv x^2+y^2-2y-3=0$ intersect each other at the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

$S_1+\lambda S_2 = 0 \Rightarrow (1+\lambda)x^2+(1+\lambda)y^2+6x+2(f-\lambda)y-3\lambda = 0$

$\therefore$ When λ is a parameter, this equation represents a circle. (5)

Since the circles $S_1 = 0$ and $S_2 = 0$ pass through the point $P_1(x_1, y_1)$,

$$x_1^2+y_1^2+6x_1+2fy_1=0 \text{ ——— } (1)$$

$$x_1^2+y_1^2-2y_1-3=0 \text{ ——— } (2)$$

From (1) + λ(2), $(1+\lambda)x_1^2+(1+\lambda)y_1^2+6x_1+2(f-\lambda)y_1-3\lambda = 0$

That is, the circle $S_1+\lambda S_2 = 0$ passes through the point $P_1(x_1, y_1)$. (10)

[see page eighteen

It can be shown similarly that the circle $S_1 + \lambda S_2 = 0$ passes through the point $P_2(x_2, y_2)$. 5

$\therefore$ $S_1 + \lambda S_2 = 0$ represents any circle that passes through the intersection points of the two circles $S_1 = 0$ and $S_2 = 0$. Here λ is a parameter.

(i) $S_1 + \lambda S_2 \equiv (1+\lambda)x^2 + (1+\lambda)y^2 + 6x + 2(f-\lambda)y - 3\lambda = 0$. Here $f = \frac{3}{2}$

When this passes through $(-2, 2)$, $(1+\lambda)4 + (1+\lambda)4 - 12 + 2(\frac{3}{2} - \lambda)(2) - 3\lambda = 0$ 10

$\Rightarrow 8 - 12 + 6 + 8\lambda - 4\lambda - 3\lambda = 0$

$\Rightarrow 2 + \lambda = 0$

$\Rightarrow \lambda = -2$

$\therefore$ The equation of the required circle is $-x^2 - y^2 + 6x + 7y + 6 = 0$

$x^2 + y^2 - 6x - 7y - 6 = 0$ 5

(ii) The equation of the common chord is given by $S_1 - S_2 = 0$

$6x + (2f + 2)y + 3 = 0$

$6x + 5y + 3 = 0$ 5

$S_1 + \lambda S_2 = (1+\lambda)x^2 + (1+\lambda)y^2 + 6x + 2(f-\lambda)y - 3\lambda = 0$

Centre $\equiv \left[-\frac{3}{1+\lambda}, \frac{\lambda - f}{1+\lambda}\right]$ 5

Since the centre of the smallest circle lies on the common chord,

$6\left(-\frac{3}{1+\lambda}\right) + 5\left(\frac{\lambda - f}{1+\lambda}\right) + 3 = 0$. Here $f = \frac{3}{2}$ 5

$-18 + 5\lambda - 5f + 3 + 3\lambda = 0$

$8\lambda = 18 - 3 + \frac{15}{2} = \frac{45}{2} \Rightarrow \lambda = \frac{45}{16}$

$\therefore$ The equation of the required circle is $\frac{61}{16}x^2 + \frac{61}{16}y^2 + 6x + 2\left(\frac{3}{2} - \frac{45}{16}\right)y - \frac{135}{16} = 0$

$61x^2 + 61y^2 + 96x - 42y - 135 = 0$ 5

75

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17. (a) $\cos A + \cos B + \cos C = \frac{3}{2}$

$1 - 2\sin^2\frac{A}{2} + 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = \frac{3}{2}$ 10

Since $A + B + C = \pi$, $2\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\frac{B-C}{2} - 2\sin^2\frac{A}{2} = \frac{1}{2}$

$2\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - \sin\frac{A}{2}\right] = \frac{1}{2}$ 5

$\cos\frac{B-C}{2} - \sin\frac{A}{2} = \frac{1}{4\sin\frac{A}{2}}$

$\Rightarrow \cos\frac{B-C}{2} = \sin\frac{A}{2} + \frac{1}{4\sin\frac{A}{2}}$

[see page nineteen

$$= \frac{4\sin^2\frac{A}{2}+1}{4\sin\frac{A}{2}} \quad (5)$$

$$= \frac{\left(1-2\sin\frac{A}{2}\right)^2+4\sin\frac{A}{2}}{4\sin\frac{A}{2}} \quad (5)$$

$$= \frac{\left(1-2\sin\frac{A}{2}\right)^2}{4\sin\frac{A}{2}}+1$$

Since $\cos\frac{B-C}{2} \le 1$, $\frac{\left(1-2\sin\frac{A}{2}\right)^2}{4\sin\frac{A}{2}}+1 \le 1$ (5)

$$\Rightarrow \frac{\left(1-2\sin\frac{A}{2}\right)^2}{\sin\frac{A}{2}} \le 0 \quad (5)$$

Since $0<A<\pi$, $\sin\frac{A}{2}>0$. $\therefore\ 1-2\sin\frac{A}{2}=0$.

Then, $\cos\frac{B-C}{2}=1$ and $\sin\frac{A}{2}=\frac{1}{2}$ (5)

$\Rightarrow B = C$ and $A = \frac{\pi}{3}$ (5) ; since $0<A<\pi$,

$\Rightarrow A = B = C = \frac{\pi}{3}$

$\Rightarrow$ Triangle ABC is equilateral.

45

(b) $f(\theta) \equiv 3\cos^2\theta+10\sin\theta\cos\theta+27\sin^2\theta$

$\equiv \frac{3}{2}[1+\cos2\theta]+5\sin2\theta+\frac{27}{2}[1-\cos2\theta]$ (5)

$\equiv 15-12\cos2\theta+5\sin2\theta$

$\equiv 15-13\left[\frac{12}{13}\cos2\theta-\frac{5}{13}\sin2\theta\right]$ (5)

$\equiv 15-13[\cos2\theta\cos\alpha-\sin2\theta\sin\alpha]$; (5) here $0<\alpha<\frac{\pi}{2}$ such that $\cos\alpha=\frac{12}{13}$ and $\sin\alpha=\frac{5}{13}$ (5)

$\equiv 15-13\cos(2\theta+\alpha)$

$\therefore f(\theta)$ is of the form $a+b\cos(2\theta+\alpha)$.

Here $a=15$, $b=-13$, (5) $\alpha=\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}\left(\frac{5}{13}\right)$ (5) . $0<\alpha<\frac{\pi}{2}$

$f(\theta) \equiv 15-13\cos(2\theta+\alpha)$; here $\alpha=\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}\left(\frac{5}{13}\right)$

$f(\theta)$ is continuous for all θ.

[see page twenty

When $\theta = 0$, $f(0) = 15 - 13\cos\alpha = 15 - 12 = 3$

When $\theta = \pi$, $f(\pi) = 15 - 13\cos(2\pi + \alpha) = 15 - 13\cos\alpha = 3$

$-1 \leq \cos(2\theta + \alpha) \leq 1 \Rightarrow -13 \leq 13\cos(2\theta + \alpha) \leq 13 \Rightarrow -13 \leq -13\cos(2\theta + \alpha) \leq 13$

5 $\Rightarrow 2 \leq 15 - 13\cos(2\theta + \alpha) \leq 28$

$\Rightarrow 2 \leq f(\theta) \leq 28$ 5

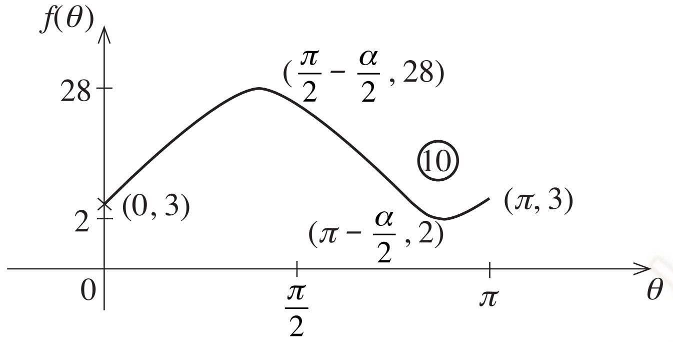
At the minimum point, $f(\theta) = 2$. Then $\cos(2\theta + \alpha) = 1 \Rightarrow 2\theta + \alpha = 2n\pi$; here $n \in \mathbb{Z}$ 5

$\theta = n\pi - \frac{\alpha}{2} = \pi - \frac{\alpha}{2}$, since $\theta \in [0, \pi]$ 5

At the maximum point, $f(\theta) = 28$. Then $\cos(2\theta + \alpha) = -1 \Rightarrow 2\theta + \alpha = 2n\pi \pm \pi$, $n \in \mathbb{Z}$ 5

$\theta = n\pi \pm \frac{\pi}{2} - \frac{\alpha}{2} = \frac{\pi}{2} - \frac{\alpha}{2}$, since $\theta \in [0, \pi]$ 5

$f(\theta) \neq 0$



The equation $f(\theta) - k = 0$

(i) has exactly one solution when, $k = 2$ and $k = 28$. 5

(ii) has two solutions when, $2 < k < 3$ and $3 < k < 28$. 5

(iii) has three solutions when $k = 3$. 5

(iv) has no solutions when, $k < 2$ and $k > 28$. 5

$\boxed{90}$

(c) $\sin^{-1}\sqrt{\frac{2}{3}} - \sin^{-1} x = \frac{\pi}{2}$ 5

Let $\alpha = \sin^{-1}\sqrt{\frac{2}{3}}$ and $\beta = \sin^{-1} x$.

Then $\alpha - \beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} + \beta$

$\Rightarrow \cos\alpha = \cos(\frac{\pi}{2} + \beta)$

$= -\sin\beta$ 5

$\Rightarrow \frac{1}{\sqrt{3}} = -x$

$\Rightarrow x = -\frac{1}{\sqrt{3}}$ 5

$\alpha = \sin^{-1}\sqrt{\frac{2}{3}}$ $\quad 0 < \alpha < \frac{\pi}{2}$

$\sin\alpha = \sqrt{\frac{2}{3}}$

$\cos\alpha = \sqrt{1 - \frac{2}{3}}$

$= \sqrt{\frac{1}{3}}$

$\boxed{15}$

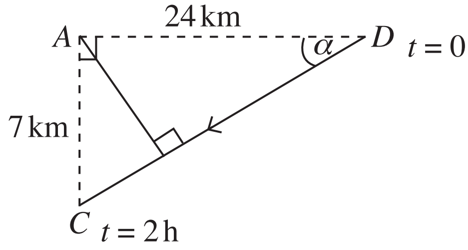
G.C.E.(A.L) Support Seminar - 2015

Combined Mathematics II

Answer Guide

Part A

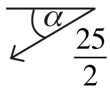
1. The path of the boat relative to the ship:



$$DC = \sqrt{24^2 + 7^2} \text{ km} = 25 \text{ km}$$

$$\tan\alpha = \frac{7}{24} \quad 5$$

S - ship, B - boat, E - earth

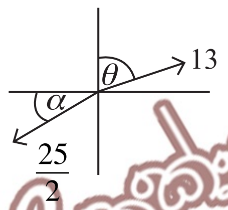
(i) $V_{B,S} =$  $\frac{25}{2}$ km h^{-1} 5

The shortest distance between the ship and the boat $= 24\sin\alpha$ km

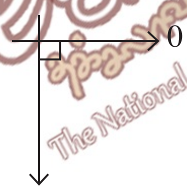
$= 24 \cdot \frac{7}{25}$ km

$= \frac{168}{25}$ km 5

(ii) $V_{B,E} = V_{B,S} + V_{S,E}$

$=$ 

5 $\quad \tan\theta = \frac{12}{5}$

$=$ 

$13\sin\theta - \frac{25}{2}\cos\alpha$

$= 13 \cdot \frac{12}{13} - \frac{25}{2} \cdot \frac{24}{25}$

$= 12 - 12$

$= 0$

$\frac{25}{2}\sin\alpha - 13\cos\theta$

$= \frac{25}{2} \cdot \frac{7}{25} - 13 \cdot \frac{5}{13}$

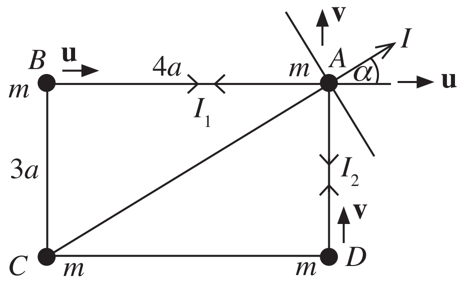
$= \frac{7}{2} - 5$

$= -\frac{3}{2}$

$\therefore V_{B,E} = \uparrow \frac{3}{2}$ km h^{-1} (To the North) 5

25

2. Since $AB \perp BC$ and $AD \perp DC$,
although the impulse applied to A generates impulses I_1 and I_2 along the strings AB and AD respectively, no impulse is generated along BC or DC. 5



Since the strings are inextensible, let us take the velocities of the particles at A and B in the direction BA as $\mathbf{u}$ and the velocities of the particles at A and D in the direction DA as $\mathbf{v}$. 5

By applying $\mathbf{I} = \Delta(m\mathbf{v})$ to the system, in the direction $\overrightarrow{BA}$, we obtain $I\cos\alpha = 2m\text{u}$

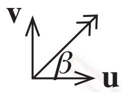
By applying $\mathbf{I} = \Delta(m\mathbf{v})$ to the system, in the direction $\overrightarrow{DA}$, we obtain $I\sin\alpha = 2m\text{v}$ 5

$\sin\alpha = \dfrac{3}{5}$ and $\cos\alpha = \dfrac{4}{5}$

$\therefore u = \dfrac{4}{5}I \cdot \dfrac{1}{2m} = \dfrac{2I}{5m}$

$v = \dfrac{3}{5}I \cdot \dfrac{1}{2m} = \dfrac{3I}{10m}$ 5

$\therefore$ The speed of $A = \dfrac{I}{10m}\sqrt{4^2+3^2} = \dfrac{I}{2m}$



$\beta = \tan^{-1}\left(\dfrac{v}{u}\right) = \tan^{-1}\left(\dfrac{3}{4}\right) = \alpha$

The speed of $B = \dfrac{2I}{5m}$ in the direction BA

The speed of $C = 0$

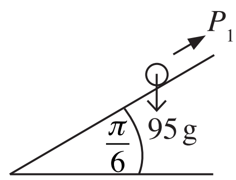
The speed of $D = \dfrac{3I}{10m}$ in the direction DA 5

25

3. By applying $Pv = H$ for the horizontal motion, $H = 3P$ ——— (1)
By applying $\mathbf{F} = m\mathbf{a}$, $P - k \cdot 3^2 = 0$ ——— (2) 5

$\Rightarrow H = 27k$ ——— (3)

For the upward motion along the inclined plane,
by applying $Pv = H$, $H = 2P_1$ ——— (4)



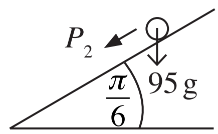
By applying $\mathbf{F} = m\mathbf{a}$, $P_1 - 95\text{g}\sin\dfrac{\pi}{6} - k \cdot 2^2 = 0$ ——— (5) 5

$\Rightarrow \dfrac{H}{2} = 475 + 4k$ ——— (6)

From (3) and (6) $\quad 19k = 950 \Rightarrow k = 50$

$H = 1350\,\text{W} = 1.35\,\text{kW}$ 5

For the downward motion along the inclined plane,
by applying $Pv = H$, $H = 4P_2$



By applying $\mathbf{F} = m\mathbf{a}$, $P_2 + 95\text{g}\sin\dfrac{\pi}{6} - 50 \cdot 4^2 = 95a$ 5

$\dfrac{1350}{4} + 475 - 800 = 95a$

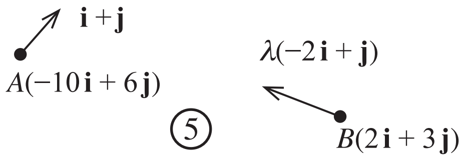
$a = \dfrac{1}{2\times 95}(675 - 650)$

$= \dfrac{25}{2\times 95} = \dfrac{5}{38}\,\text{m s}^{-2}$ 5

25

[see page three

4. When $t = 0$



If the two particles collide when $t = T$,
the displacement of $P = (\mathbf{i}+\mathbf{j})T$
the displacement of $Q = (-2\mathbf{i}+\mathbf{j})\lambda T$
By considering the position of the point at which the collision occurs

$$-10\mathbf{i}+6\mathbf{j}+(\mathbf{i}+\mathbf{j})T \equiv 2\mathbf{i}+3\mathbf{j}+(-2\mathbf{i}+\mathbf{j})\lambda T$$
5 5

$\Rightarrow \quad (-10+T-2+2\lambda T)\mathbf{i} = (3+\lambda T-6-T)\mathbf{j}$

Since $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{i}, \mathbf{j} \neq \mathbf{0}$

$2\lambda T + T = 12$ and $\lambda T - T = 3$

$\Rightarrow 3T = 6$

$\Rightarrow T = 2 \qquad \therefore 2\lambda = 5$

$\lambda = \frac{5}{2}$ 5

$\therefore \mathbf{v} = \frac{5}{2}(-2\mathbf{i}+\mathbf{j}) \Rightarrow |\mathbf{v}| = \frac{5}{2}\sqrt{(-2)^2+1^2} = \frac{5\sqrt{5}}{2}$ 5

$\boxed{25}$

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5. $\overrightarrow{OA} = \mathbf{a}+2\mathbf{b}$, $\overrightarrow{OB} = 3\mathbf{a}-\mathbf{b}$

$OA \perp OB \quad \Rightarrow \overrightarrow{OA}\,.\,\overrightarrow{OB} = 0$ 5

$\Rightarrow (\mathbf{a}+2\mathbf{b}).(3\mathbf{a}-\mathbf{b}) = 0$

$\Rightarrow 3|\mathbf{a}|^2 + 5\,\mathbf{a}\,.\,\mathbf{b} - 2|\mathbf{b}|^2 = 0$, 5 since $\mathbf{a}\,.\,\mathbf{b} = \mathbf{b}\,.\,\mathbf{a}$

$\Rightarrow \mathbf{a}\cdot\mathbf{b} = \frac{2}{5}|\mathbf{b}|^2 - \frac{3}{5}|\mathbf{a}|^2$ 5

If $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 1$, $\mathbf{a}\,.\,\mathbf{b} = \frac{2}{5} - \frac{12}{5} = -2$

$|\mathbf{a}|\,|\mathbf{b}|\cos\theta = -2$ 5 Here θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. $0 \leq \theta \leq \pi$

$2\cos\theta = -2$

$\cos\theta = -1$

$\theta = \pi$ 5

$\boxed{25}$

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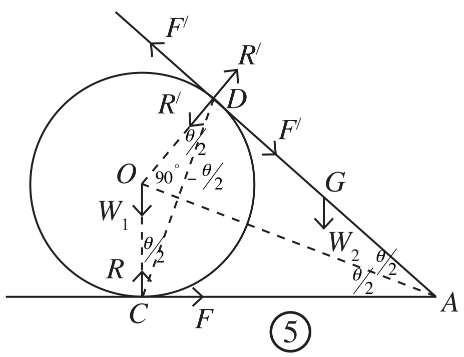
6. The three forces being concurrent. 5



Since the weight of the cylinder and the force applied on the cylinder by the rough horizontal plane pass through the point C, the force applied on the cylinder at the point D by the rod should also pass through C. 5

Then, $O\hat{C}D = O\hat{D}C = \frac{\theta}{2}$

For the equilibrium of the cylinder $\left|\frac{F}{R}\right| \leq \mu$. 5

$\therefore \tan\frac{\theta}{2} \leq \tan\lambda$

$\Rightarrow \lambda \geq \frac{\theta}{2}$, (Since $0 < \lambda, \theta < \frac{\pi}{2}$)
5

$\boxed{25}$

[see page four

7. Let A, B and C be respectively the events of the three children A, B and C independently solving the problem correctly.

Then $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{2}$ and $P(C) = \frac{1}{3}$.

Let X be the event of exactly two children independently solving the problem correctly.

$$\text{Then, } X = (A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$$

$$P(X) = P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A' \cap B \cap C) \quad (\text{Axiom}) \; 5$$

$$= P(A)\,P(B)\,P(C') + P(A)\,P(B')\,P(C) + P(A')\,P(B)\,P(C) \quad 5$$

(Since A, B, C are independent of each other)

$$= \frac{1}{6}\,\frac{1}{2}\,(1 - \frac{1}{3}) + \frac{1}{6}\,(1 - \frac{1}{2})\frac{1}{3} + (1 - \frac{1}{6})\,\frac{1}{2}\,\frac{1}{3} \quad 10$$

$$= \frac{1}{6}\,\frac{1}{2}\,\frac{2}{3} + \frac{1}{6}\,\frac{1}{2}\,\frac{1}{3} + \frac{5}{6}\,\frac{1}{2}\,\frac{1}{3}$$

$$= \frac{8}{36} \quad 5$$

$$= \frac{2}{9}$$

$\boxed{25}$

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8. R : Obtaining a plant with pink flowers

$P(R) = \frac{1}{6}$ and $P(R') = 1 - \frac{1}{6} = \frac{5}{6}$ 5

n : Number of seeds that need to be planted

X : Obtaining at least one plant with pink flowers

$P(X) = 1 - P$(Not obtaining even one plant with pink flowers)

$$= 1 - P(R' \cap R' \dots \cap R') \quad 5$$

$$= 1 - \{P(R')\}^n > 0.98 \quad 5$$

$$0.02 > \left(\frac{5}{6}\right)^n$$

$$n > \frac{\ln(0.02)}{\ln\left(\frac{5}{6}\right)} = 21.46 \quad 5$$

$$\therefore n_{\text{minimum}} = 22 \quad 5$$

$\boxed{25}$

9.

x	-2	-1	0	1	2
f	4	1	3	1	1
$f.x$	-8	-1	0	1	2
$f.x^2$	16	1	0	1	4

5

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{-6}{10} = -0.6 \quad 5$$

$$\sigma_x^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{22}{10} - 0.36 = 1.84 \quad 5$$

$$\therefore \sigma_x = \sqrt{1.84}$$

Let $y = 2000 - 4x$

Then $\bar{y} = 2000 - 4\bar{x}$

$= 2000 + 2.4$

$= 2002.4$ 5

$$\sigma_y^2 = 4^2 \cdot \sigma_x^2$$

$$= 16 \times 1.84$$

$$\therefore \sigma_y = 4\sqrt{1.84} \quad 5$$

$\boxed{25}$

10. $\bar{x} = \frac{\sum x_i}{20} = 40$

$\therefore$ The sum of all the marks $= 20 \times 40 = 800$

The sum of the least **six** marks $= 6 \times 25 = 150$ 5

The sum of the highest **six** marks $= 440$

$\therefore$ The sum of the marks of the remaining 8 children

$= 800 - 590 = 210$ 5

(i) $\therefore$ The mean of the marks of the remaining 8 children

$= \frac{210}{8} = 26.25$ 5

(ii) The third quartile $= Q_3$

$\frac{3}{4}(n+1) = \frac{3}{4} \times 21 = 15\frac{3}{4}$

The 15th mark $= 70$ 5

The 16th mark $= 71$

$\therefore Q_3 = 70 + \frac{3}{4}(71 - 70)$

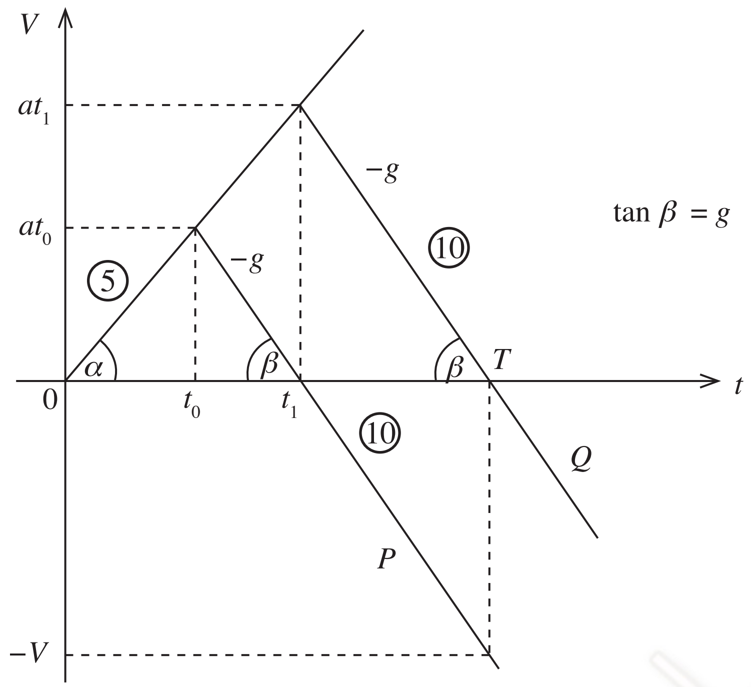
$= 70.75$ 5

$\boxed{25}$

[see page six

Part B

11. (a)



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For the motion of the particle P,

$$\tan\beta = \frac{at_0}{t_1 - t_0} \quad (5) \qquad \Rightarrow \qquad g = \frac{at_0}{t_1 - t_0} \quad (5)$$

$$t_1 - t_0 = \frac{at_0}{g}$$

$$t_1 = \frac{at_0}{g} + t_0$$

$$t_1 = \frac{t_0}{g}(a + g) \quad (5)$$

If the time taken for Q to come to instantaneous rest is T,

$$\tan\beta = \frac{at_1}{T - t_1} \qquad \Rightarrow \qquad g = \frac{at_1}{T - t_1} \Rightarrow T - t_1 = \frac{at_1}{g} \quad (5)$$

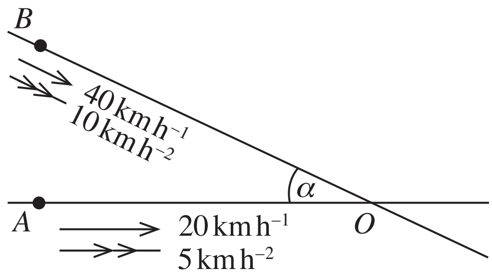
$$\tan\beta = \frac{V}{T - t_1}$$

$$V = g(T - t_1)$$

$$= \cancel{g}\frac{at_1}{\cancel{g}} = at_0\left(\frac{a}{g} + 1\right) \quad (5)$$

25

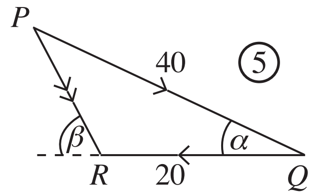
(b)



$V(Y, X) = V(Y, E) + V(E, X)$

$= \underset{\alpha}{\searrow} 40\,\text{km h}^{-1} + \overleftarrow{\quad} 20\,\text{km h}^{-1}$ 5

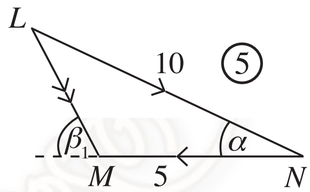
$= \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$



and $a(Y, X) = a(Y, E) + a(E, X)$

$= \underset{\alpha}{\searrow} 10\,\text{km h}^{-2} + \overleftarrow{\quad} 5\,\text{km h}^{-2}$ 5

$= \overrightarrow{LN} + \overrightarrow{NM} = \overrightarrow{LM}$



(i) $\tan\beta = \dfrac{40\sin\alpha}{40\cos\alpha - 20}$ 5

$= \dfrac{2\sin\alpha}{2\cos\alpha - 1}$

(ii) $\tan\beta_1 = \dfrac{10\sin\alpha}{10\cos\alpha - 5} = \dfrac{2\sin\alpha}{2\cos\alpha - 1}$ 5

From (i) and (ii), $\tan\beta = \tan\beta_1$

$\beta = \beta_1 \; (0 < \beta, \beta_1 < \pi)$ 5

$\Rightarrow V(Y, X) \parallel a(Y, X)$ 5

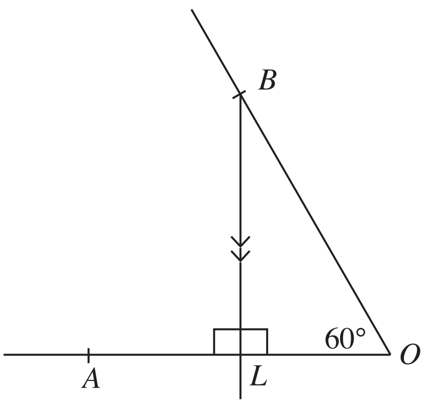
$\therefore$ The path of Y relative to X is a straight line.

$\boxed{40}$

If $\alpha = 60°$, then $\tan\beta, \tan\beta_1$ are not defined.

$\therefore \beta = \beta_1 = 90°$ 5

$\therefore$ The path of Y relative to X is perpendicular to OA.



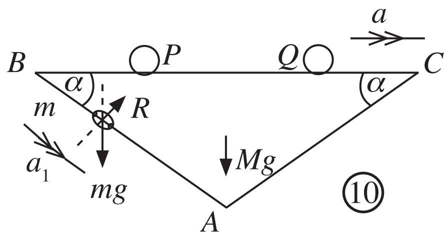
The shortest distance between X and Y

$= AL = 10 - 10\cos 60°$

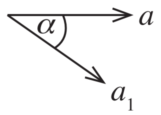
$= 5\,\text{km}$ 5

$\boxed{10}$

(c)



$a(M, E) = \twoheadrightarrow a$, $a(m, E) =$



5

By applying $\mathbf{F} = m\mathbf{a}$

m : ; $mg\sin\alpha = m(a_1 + a\cos\alpha)$ ——— (1) 5

$M, m \longrightarrow$; $0 = Ma + m(a + a_1\cos\alpha)$

$0 = (M+m)a + m\,a_1\cos\alpha$ ——— (2) 10

From (1) and (2) ;

$0 = (M+m)a + m(g\sin\alpha - a\cos\alpha)\cos\alpha$ 5

$-mg\sin\alpha\cos\alpha = (M + m - m\cos^2\alpha)\,a$

$$\therefore a = -\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha} = -\frac{mg\sin 2\alpha}{2(M + m\sin^2\alpha)}$$ 5

m : ; $R - mg\cos\alpha = ma\sin\alpha$ 5

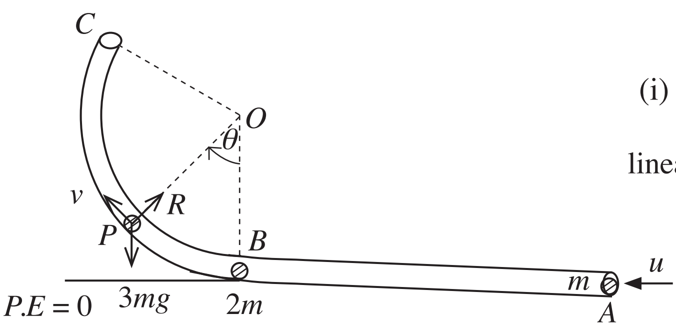
$$R = mg\cos\alpha - \frac{m^2 g\sin^2\alpha\cos\alpha}{M + m\sin^2\alpha}$$

$$= \frac{mg\cos\alpha}{M + m\sin^2\alpha}\{M + \cancel{m\sin^2\alpha} - \cancel{m\sin^2\alpha}\}$$ 5

$$= \frac{Mmg\cos\alpha}{M + m\sin^2\alpha}$$

$\boxed{50}$

12. (a)



(i)

By the law of conservation of linear momentum;

$3mv_1 = mu$ 5

$v_1 = \dfrac{u}{3}$ 5

$\boxed{10}$

[see page nine

(ii) By the law of conservation of energy

$$\frac{1}{2}(3m)v^2 + 3mg(a - a\cos\theta) = \frac{1}{2}(3m)\frac{u^2}{9} \quad 15$$

$$v^2 = \frac{u^2}{9} - 2ga(1 - \cos\theta) \quad 5$$

Applying **F** = m**a**

$$\overrightarrow{PO};\ R - 3mg\cos\theta = 3m\left(\frac{v^2}{a}\right) \quad 10$$

$$R = \frac{3m}{a}\left\{\frac{u^2}{9} - 2ga + 3ga\cos\theta\right\} \quad 5$$

$\boxed{35}$

(iii) When $\theta = \frac{2\pi}{3}$; $v^2 = \frac{u^2}{9} - 2ga\left(1+\frac{1}{2}\right) = \frac{u^2}{9} - 3ga$ 5

5 If $\frac{u^2}{9} - 3ga > 0$ the particle leaves at C ; i.e., when $u > 3\sqrt{3ga}$

$\boxed{10}$

(iv) Applying $s = ut + \frac{1}{2}at^2$ for the motion from C to A,

$$\rightarrow \sqrt{3}\,a + \frac{\sqrt{3}a}{2} = v\left(\frac{1}{2}\right)t \quad \Rightarrow 3\sqrt{3}\,a = vt \text{ —— (1)} \quad 5$$

$$\uparrow -\frac{3a}{2} = v\frac{\sqrt{3}}{2}t - \frac{1}{2}gt^2 \text{ —— (2)} \quad 5$$

From (1) and (2),

$$-\frac{3a}{2} = v\frac{\sqrt{3}}{2}\cdot\frac{3\sqrt{3}a}{v} - \frac{1}{2}g\frac{27}{v^2}a^2 \quad 5$$

$$\frac{1}{2}ga\frac{27}{v^2} = \frac{9}{2} + \frac{3}{2}$$

$$27ga = 12v^2$$

$$\frac{9}{4}ga = \frac{u^2}{9} - 3ga$$

$$\frac{u^2}{9} = \frac{9}{4}ga + 3ga \quad 5$$

$$\therefore\ u = \frac{3}{2}\sqrt{21ag}$$

$\boxed{20}$

(b) By the law of conservation of linear momentum

$$mv_1 + Mv_2 = mu \text{ —— (i)} \quad 10$$

By Newton's law of restitution $v_1 - v_2 = -eu$ —— (ii) 10

From (i) and (ii) $v_1 = \frac{u(m - eM)}{(m+M)}$ and $v_2 = \frac{mu(1+e)}{(m+M)}$ 10

[see page ten

Applying $\Delta E = \sum \frac{1}{2} I.(U+V)$ 5

$= \frac{1}{2}(I)(v_2) + \frac{1}{2}(-I)(u+v_1) = -\frac{I}{2}(u+v_1-v_2) = -\frac{I}{2}(u-eu)$ 10

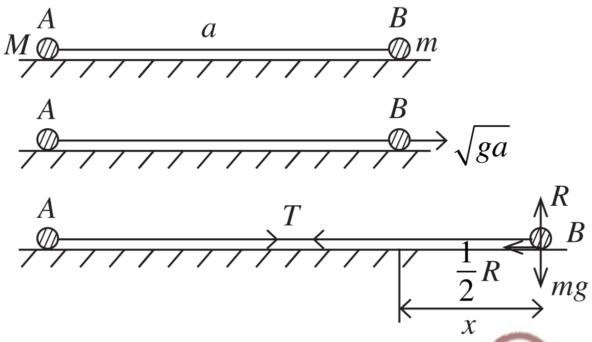
$= -\frac{I}{2}(1-e)u$ 5 ; where $I = Mv_2 = \frac{Mmu(1+e)}{(M+m)}$ 5

$\frac{1}{2}\frac{Mmu^2(1+e)(1-e)}{(M+m)} = \frac{1}{4}mu^2$ 10

$M(1-2e^2) = m$ 5 $\Rightarrow$ $1-2e^2 > 0$ 5 $\Rightarrow$ $e < \frac{1}{\sqrt{2}}$

75

13.



By applying $\uparrow$ $\mathbf{F} = m\mathbf{a}$ to the particle B, $R = mg$

By applying $\longrightarrow$ $\mathbf{F} = m\mathbf{a}$ to the particle B, $-\frac{1}{2}R - T = m\ddot{x}$ 10

Since the modulus of elasticity of the string is $2mg$, $T = 2mg\frac{x}{a}$ 10

$\Rightarrow \quad -\frac{1}{2}mg - \frac{2mgx}{a} = m\ddot{x}$ 10

$\Rightarrow \quad \ddot{x} = -\frac{2g}{a}\left(x+\frac{a}{4}\right)$ ———— (1)

$\therefore$ the motion of the particle is a simple harmonic motion with the centre of oscillation given by $x = -\frac{a}{4}$ 5

By taking $x + \frac{a}{4} = \alpha\cos\omega t + \beta\sin\omega t$, ———— (2)

since $x = 0$ when $t = 0$, $\alpha = \frac{a}{4}$ 5

By differentiating with respect to t, $\dot{x} = -\alpha\omega\sin\omega t + \beta\omega\cos\omega t$ ———— (3)

Since $\dot{x} = \sqrt{ga}$ when $t = 0$, $\sqrt{ga} = \beta\omega \Rightarrow \beta = \frac{\sqrt{ga}}{\omega}$ 5

By differentiating again with respect to t,

$$\begin{aligned}\ddot{x} &= -\alpha\omega^2\cos\omega t - \beta\omega^2\sin\omega t \\ &= -\omega^2(\alpha\cos\omega t + \beta\sin\omega t) \\ &= -\omega^2\left(x + \frac{a}{4}\right)\end{aligned}$$ 5

[see page eleven

From (1), $\omega^2 = \dfrac{2g}{a} \Rightarrow \omega = \sqrt{\dfrac{2g}{a}}$ 5

$\therefore \alpha = \dfrac{a}{4}$ and $\beta = \sqrt{ga}\,\dfrac{\sqrt{a}}{\sqrt{2g}} = \dfrac{a}{\sqrt{2}}$ 5 $\boxed{60}$

Since $\dot{x} = 0$ at the maximum extension, 10
from (3), $\alpha \sin \omega t = \beta \cos \omega t$

$$\Rightarrow \tan \omega t = \frac{\beta}{\alpha} = \frac{a/\sqrt{2}}{a/4} = 2\sqrt{2} \quad 5$$

From (2), $x + \dfrac{a}{4} = \dfrac{a}{4}\,\dfrac{1}{3} + \dfrac{a}{\sqrt{2}}\,\dfrac{2\sqrt{2}}{3}$

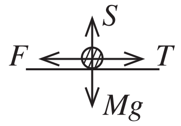
$$= \frac{a}{12}(1+8)$$

$$= \frac{3a}{4}$$

$$\Rightarrow \quad x = \frac{a}{2} \quad 5$$

$\therefore$ The maximum extension is $\dfrac{a}{2}$. $\boxed{20}$

The maximum tension $= \dfrac{2mg}{a}\,\dfrac{a}{2} = mg$ 5

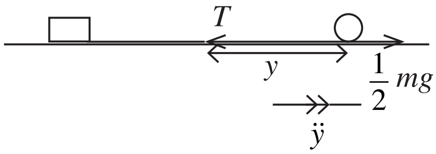


For the equilibrium of particle A,

$S = Mg$ and $T = F$

Since $\dfrac{F}{S} \leq \dfrac{1}{2}$, 10 $\quad \dfrac{mg}{Mg} \leq \dfrac{1}{2}$ 5

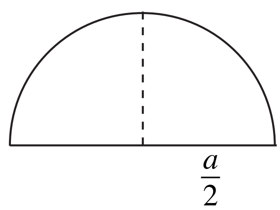
$\therefore \quad M \geq 2m$ $\boxed{20}$



$$-T + \frac{1}{2}mg = m\ddot{y} \quad 5$$

$$-\frac{2mg}{a}y + \frac{mg}{2} = m\ddot{y}$$

$$\ddot{y} = -\frac{2g}{a}\left(y - \frac{a}{4}\right) \quad 5$$



$\boxed{10}$

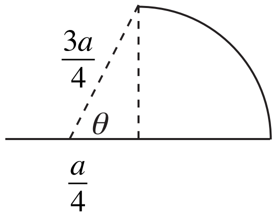
[see page twelve

Centre of oscillation $= \dfrac{a}{4}$ (5)

Angular velocity $= \sqrt{\dfrac{2g}{a}}$ (5)

By considering the circular motion corresponding to the simple harmonic motion :

Time taken to return $= \pi\sqrt{\dfrac{a}{2g}}$ (5)



Time taken to reach maximum extension

$$= \frac{\theta}{\omega} = \sqrt{\frac{a}{2g}}\cos^{-1}\frac{1}{3} \quad (5)$$

$\therefore$ Total time $=$

$$\sqrt{\frac{a}{2g}}\cos^{-1}\frac{1}{3} + \pi\sqrt{\frac{a}{2g}}$$

$$= \left[\pi + \cos^{-1}\left(\frac{1}{3}\right)\right]\sqrt{\frac{a}{2g}} \quad (5)$$

For the return motion, $\dot{y} = \dfrac{a}{4}\left(-\sqrt{\dfrac{2g}{a}}\right)\sin\sqrt{\dfrac{2g}{a}}\,t$ (5)

When $t = \pi\sqrt{\dfrac{a}{2g}}$

$$\sqrt{\frac{2g}{a}}\,t = \pi$$

$$\therefore \quad \dot{y} = 0 \quad (5)$$

$\therefore$ The particle B returns to the initial point and comes to a definite rest. (5)

$\boxed{40}$

Aliter :

For the motion of particle B :

Let the time for particle B to reach the maximum extension be t_0.

When $\tan \omega t = 2\sqrt{2}$, then $\cos \omega t = \dfrac{1}{3}$ (5)

$$\therefore \cos \omega t_0 = \frac{1}{3}$$

$$\omega t_0 = \cos^{-1}\left(\frac{1}{3}\right)$$

$$t_0 = \frac{1}{\omega}\cos^{-1}\left(\frac{1}{3}\right) = \sqrt{\frac{a}{2g}}\cos^{-1}\left(\frac{1}{3}\right) \quad (5)$$

For the return motion of particle B :

Let the time taken to reach the initial point be t_1.

Then $y = 0$. (5)

$$\therefore \cos\sqrt{\frac{2g}{a}}\,t_1 = -1$$

[see page thirteen

$$\sqrt{\frac{2g}{a}}\,t_1 = \pi \quad \Rightarrow \quad t_1 = \sqrt{\frac{a}{2g}}\,\pi \qquad 5$$

$\therefore$ The time taken for particle B to reach the initial point

$$= t_0 + t_1$$

$$= \sqrt{\frac{a}{2g}}\left[\pi + \cos^{-1}\left(\frac{1}{3}\right)\right] \qquad 5$$

If the velocity when $t_1 = \sqrt{\frac{a}{2g}}\,\pi$ is $\dot{y}$,

$$\dot{y} = -\frac{a}{4}\sqrt{\frac{2g}{a}}\,\sin\sqrt{\frac{2g}{a}}\,t_1 \qquad 5$$

$$= -\frac{a}{4}\sqrt{\frac{2g}{a}}\,\sin\sqrt{\frac{2g}{a}}\sqrt{\frac{a}{2g}}\,\pi$$

$$= 0 \qquad 5$$

$\therefore$ Particle B comes to rest definitely at the initial point 5

$\boxed{40}$

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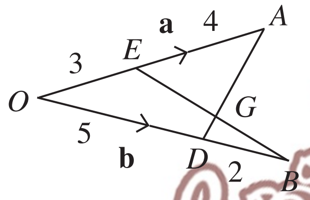
14. (a)



$$\overrightarrow{OG} = \overrightarrow{OB} + \overrightarrow{BG}$$
$$= \overrightarrow{OB} + \lambda\overrightarrow{BE}$$
$$= \overrightarrow{OB} + \lambda[\overrightarrow{BO} + \overrightarrow{OE}]$$
$$= \overrightarrow{OB} + \lambda(-\overrightarrow{OB} + \frac{3}{7}\overrightarrow{OA})$$
$$= \mathbf{b} + \lambda(\frac{3}{7}\mathbf{a} - \mathbf{b}) \text{ ——— (1)} \qquad 15$$

$$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG}$$
$$= \overrightarrow{OA} + \mu\overrightarrow{AD}$$
$$= \overrightarrow{OA} + \mu[\overrightarrow{AO} + \overrightarrow{OD}]$$
$$= \overrightarrow{OA} + \mu(-\overrightarrow{OA} + \frac{5}{7}\overrightarrow{OB})$$
$$= \mathbf{a} + \mu(\frac{5}{7}\mathbf{b} - \mathbf{a}) \text{ ——— (2)} \qquad 15$$

From (1) and (2),

$$\mathbf{b} + \lambda(\frac{3}{7}\mathbf{a} - \mathbf{b}) = \mathbf{a} + \mu(\frac{5}{7}\mathbf{b} - \mathbf{a}) \qquad 5$$

Since **a** and **b** are independent vectors,

$$\frac{3\lambda}{7} = 1 - \mu \text{ ——— (3)} \qquad 5$$

$$1 - \lambda = \frac{5\mu}{7} \text{ ——— (4)} \qquad 5$$

$(3) + \frac{3}{7} \times (4)$

$$\frac{3}{7} = 1 - \mu + \frac{15}{49}\mu$$

$$\frac{\cancel{4}\,2}{\cancel{7}\,1} = \frac{17\,\cancel{34}}{7\,\cancel{49}}\mu$$

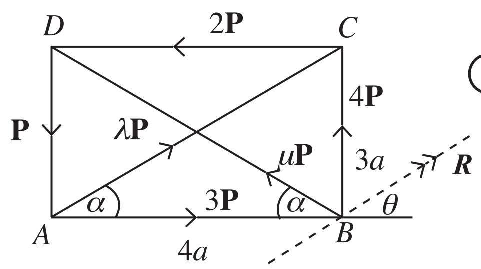
$$\Rightarrow \quad \mu = \frac{14}{17} \qquad 5$$

$$\overrightarrow{OG} = \mathbf{a} + \frac{14}{17}\left(\frac{5}{7}\mathbf{b} - \mathbf{a}\right) = \mathbf{a} + \frac{10}{17}\mathbf{b} - \frac{14}{17}\mathbf{a}$$

$$= \frac{3}{17}\mathbf{a} + \frac{10}{17}\mathbf{b} = \frac{1}{17}(3\mathbf{a} + 10\mathbf{b}) \qquad 10$$

$\boxed{60}$

(b)



5

(i) $\boldsymbol{R} = 0$ when the system is equivalent to a couple. 5

Then $\longrightarrow \boldsymbol{X} = 0, \quad \uparrow \boldsymbol{Y} = 0.$

$$\longrightarrow \boldsymbol{X}: \quad 3P + \lambda P\cos\alpha - \mu P\cos\alpha - 2P = 0 \qquad 5$$

$$3P + \lambda P\frac{4}{5} - \mu P\frac{4}{5} - 2P = 0$$

$$5\cancel{P} + 4\lambda\cancel{P} - 4\mu\cancel{P} = 0$$

$$5 = 4\mu - 4\lambda \text{ ——— (1)} \qquad 5$$

$$\uparrow \boldsymbol{Y}: \quad 4P - P + \lambda P\sin\alpha + \mu P\sin\alpha = 3P + \lambda P\frac{3}{5} + \mu P\frac{3}{5} = 0 \qquad 5$$

$$\lambda + \mu = -5 \text{ ——— (2)} \qquad 5$$

(1) + (2) × 4

$$8\mu = -15 \Rightarrow \mu = -\frac{15}{8} \qquad 5$$

$$\lambda = -5 - \mu = -5 + \frac{15}{8} = -\frac{25}{8} \qquad 5$$

$\boxed{40}$

(ii) $\circlearrowleft B = 0$ 5

$$2\cancel{P} \times 3\cancel{a} + \cancel{P} \times 4\cancel{a} - \lambda\cancel{P} \times 4\cancel{a}\sin\alpha = 0$$

$$6 + 4 - \lambda \times 4 \times \frac{3}{5} = 0 \Rightarrow \lambda = \frac{\cancel{50}\ 25}{\cancel{12}\ 6} = \frac{25}{6} \qquad 5$$

$$\longrightarrow R\cos\theta = P + (\lambda - \mu)P\cos\alpha \text{ ——— (1)} \qquad 5$$

$$\uparrow R\sin\theta = 3P + (\lambda + \mu)P\sin\alpha \text{ ——— (2)} \qquad 5$$

From (2)/(1), $\quad \tan\theta = \dfrac{3P + (\lambda+\mu)P\times\frac{3}{5}}{P + (\lambda-\mu)P\times\frac{4}{5}}$ 5

[see page fifteen

Since R is parallel to AC, $\tan\theta = \tan\alpha$

$$\theta = \alpha \quad 5$$

$$\frac{3}{4}\ 5 = \frac{3+\left(\frac{25}{6}+\mu\right)\times\frac{3}{5}}{1+\left(\frac{25}{6}-\mu\right)\times\frac{4}{5}} = \frac{3+\frac{5}{2}+\frac{3\mu}{5}}{1+\frac{5\times 2}{3}-\frac{4\mu}{5}}$$

$$13-\frac{12\mu}{5} = 22+\frac{12\mu}{5} \Rightarrow 9 = -\frac{24\mu}{5} \Rightarrow \mu = -\frac{45}{24} \quad 5$$

For equilibrium, $\boldsymbol{R} = 0$ and the moment about any point should be zero.
From parts (i) and (ii), it is clear that it is not possible to find λ, μ satisfying both these conditions simultaneously. 10

$\boxed{50}$

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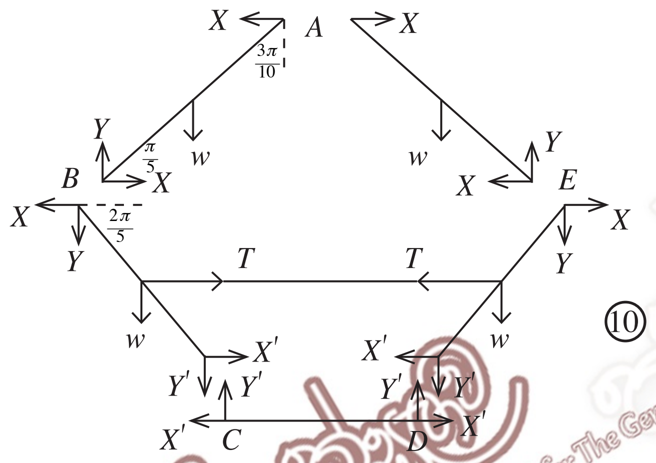
15. (a)



10

Let the length of a rod be $2a$. The system is symmetric about the vertical line through A.

$\therefore\ Y = 0$ at A. 5

For the equilibrium of the rod AB,

$$B \curvearrowleft \quad X\times 2a\sin\frac{\pi}{5} - w\times a\cos\frac{\pi}{5} = 0 \quad 10$$

$$X = \frac{w}{2}\cot\frac{\pi}{5} \quad 5$$

For the equilibrium of the rod $AB\ \uparrow$,

$$\uparrow \quad Y = w \quad 5$$

For the equilibrium of the rod BC,

$$C \curvearrowleft \quad X\times 2a\sin\frac{2\pi}{5} + Y\times 2a\cos\frac{2\pi}{5} + w\times a\cos\frac{2\pi}{5} - T\times a\sin\frac{2\pi}{5} = 0 \quad 10$$

$$T = 2X + Y\,2\cot\frac{2\pi}{5} + w\cot\frac{2\pi}{5}$$

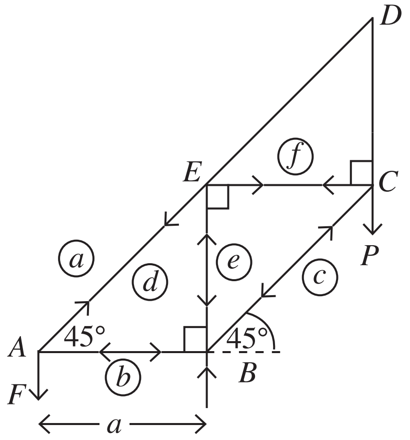
$$= \cancel{2}\,\frac{w}{\cancel{2}}\cot\frac{\pi}{5} + 2w\cot\frac{2\pi}{5} + w\cot\frac{2\pi}{5} \quad 5$$

$$= w\left(\cot\frac{\pi}{5} + 3\cot\frac{2\pi}{5}\right) \quad 10$$

$\boxed{60}$

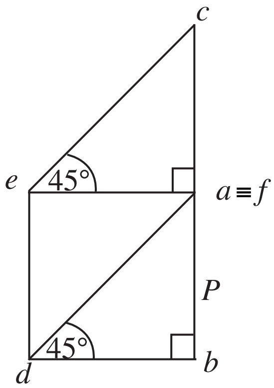
[see page sixteen

(b)



$\overset{\curvearrowleft}{B}$; $F \times a - P \times a = 0$

$F = P\text{N}$ 10



30

$a \equiv f$

a and f are concurrent. 5

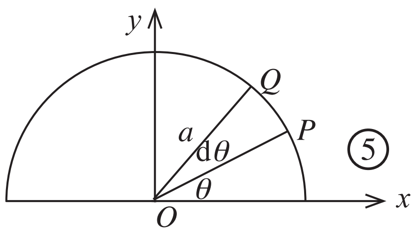
Rod	Magnitude	Stress
AB	P N	Thrust
BC	$\sqrt{2}\,P$ N	Thrust
CD	0	–
DE	0	–
AE	$\sqrt{2}\,P$ N	Tension
BE	P N	Thrust
EC	P N	Tension

35

The reaction on the support $\boldsymbol{R} = 2P$ N 10

$\boxed{90}$

16.



5

Due to symmetry about Oy, the centre of mass lies on Oy. 5
Let ρ be the mass of a unit length. The radius of the wire frame is a.

The mass of the elemental arc $PQ = \rho\, a\, d\theta$

The distance from Ox to the centre of mass of PQ is $y = a\sin\theta$

If the distance from O to the centre of mass of the object is $\bar{y}$,

$$\bar{y} = \frac{\int y\,dm}{\int dm}$$

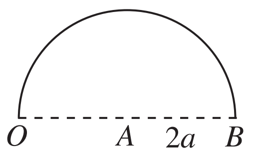
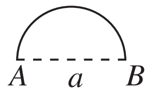
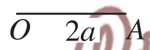
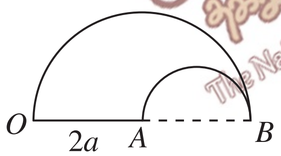
$$= \frac{\int_0^{\pi} \rho a \sin\theta a \, d\theta}{\int_0^{\pi} \rho a \, d\theta} \quad 10 \quad = \frac{a^2 \rho \int_0^{\pi} \sin\theta \, d\theta}{a\rho \int_0^{\pi} d\theta}$$

$$= \frac{a^2 \rho \left[-\cos\theta\right]_0^{\pi}}{a\rho \left[\theta\right]_0^{\pi}} = \frac{a\left[-\cos\pi + \cos 0\right]}{\pi} \quad 10$$

$$= \frac{2a}{\pi}$$

$\boxed{30}$

Let ρ be the mass of a unit length.

Object	**Mass**	**Distance from the x axis**	**Distance from the y axis**
	$2\pi a \rho$	$\frac{4a}{\pi}$	$2a$
	$\pi a \rho$	$\frac{2a}{\pi}$	$3a$
	$2a\rho$	0	a
	$3\pi a \rho + 2a\rho$	$\bar{y}$	$\bar{x}$

30

$$\bar{x} = \frac{2a \times 2\pi a\rho + 3a \times \pi a\rho + 2a\rho \times a}{3\pi a\rho + 2a\rho} \quad 10$$

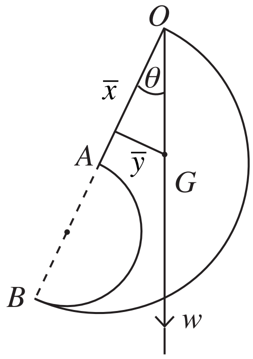
$$= \frac{4\pi a + 3\pi a + 2a}{3\pi + 2} = \frac{7\pi a + 2a}{3\pi + 2} \quad 5$$

$$\bar{y} = \frac{\frac{4a}{\pi} \times 2\pi a\rho + \frac{2a}{\pi} \times \pi a\rho}{3\pi a\rho + 2a\rho} \quad 10 \quad = \frac{10a}{3\pi + 2} \quad 5$$

$$G(\bar{x}, \bar{y}) = \left[\frac{(7\pi + 2)a}{3\pi + 2}, \frac{10a}{3\pi + 2}\right]$$

$\boxed{60}$

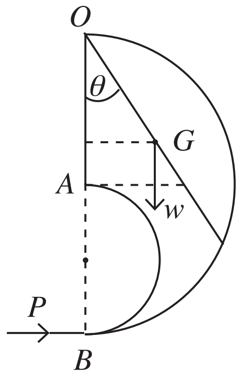
[see page eighteen



5

$$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{\frac{10a}{3\pi+2}}{\frac{(7\pi+2)a}{3\pi+2}} \; (10) \; = \frac{10}{7\pi+2} \quad (5)$$

$\boxed{20}$



5

By taking moments about O,

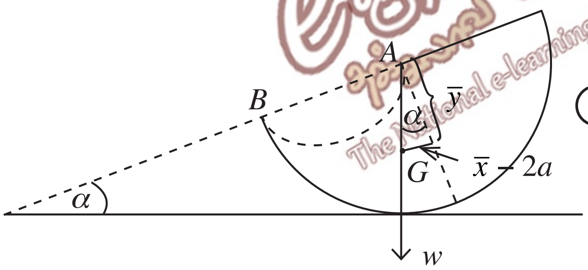
$\overset{\curvearrowleft}{O}$;

$$P \times 4a = w \times \bar{y} \quad (10)$$

$$P = \frac{w}{4a} \times \frac{10a}{3\pi+2}$$

$$= \frac{5w}{2(3\pi+2)} \quad (5) \text{ units}$$

$\boxed{20}$



5

$$\tan\alpha = \frac{\bar{x}-2a}{\bar{y}}$$

$$= \frac{\frac{(7\pi+2)a}{3\pi+2} - 2a}{\frac{10a}{3\pi+2}} \quad (10)$$

$$= \frac{(7\pi+2)\cancel{a} - 2\cancel{a}(3\pi+2)}{10\cancel{a}}$$

$$\tan\alpha = \frac{\pi-2}{10}$$

$$\alpha = \tan^{-1}\left(\frac{\pi-2}{10}\right) \quad (5)$$

$\boxed{20}$

[see page nineteen

17. (a) $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$; $P(B) > 0$ 5 $\boxed{5}$

$$\begin{aligned}
P(A_1\cap A_2\cap A_3) &= P[(A_1\cap A_2)\cap A_3] \\
&= P(X\cap A_3) \ ; \ \text{Here } X = A_1\cap A_2. \\
&= P(X).\,P(A_3|X) \quad 5 \\
P(X) &= P(A_1\cap A_2) \\
&= P(A_1).\,P(A_2|A_1) \quad 5
\end{aligned}$$

$\therefore\ P(A_1\cap A_2\cap A_3) = P(A_1).\,P(A_2|A_1).\,P(A_3|A_1\cap A_2)$ $\boxed{10}$

A : The selected member being an adult
C : The selected member being a child
F : The selected member being a female
M : The selected member being a male
S : The selected member being one who uses the swimming pool.

$P(A) = \frac{3}{4}$ $\qquad P(C) = \frac{1}{4}$

$P(M|A) = \frac{3}{4}$ $\qquad P(F|A) = \frac{1}{4}$

$P(M|C) = \frac{3}{5}$ $\qquad P(F|C) = \frac{2}{5}$

$P(S|A\cap M) = \frac{1}{2}$ $\qquad P(S|A\cap F) = \frac{1}{3}$

$P(S|C\cap M) = \frac{4}{5}$ $\qquad P(S|C\cap F) = \frac{4}{5}$ $\quad\Rightarrow\quad P(S|C) = \frac{4}{5}$ 10

(i)
$$\begin{aligned}
P(S) &= P(A\cap M\cap S) + P(A\cap F\cap S) + P(C\cap S) \quad 5 \\
&= P(A)\,P(M|A)\,P(S|A\cap M) + P(A)\,P(F|A)\,P(S|A\cap F) + P(C)\,P(S|C) \quad 5 \\
&= \frac{3}{4}\times\frac{3}{4}\times\frac{1}{2} + \frac{3}{4}\times\frac{1}{4}\times\frac{1}{3} + \frac{1}{4}\times\frac{4}{5} \quad 5 \\
&= \frac{9}{32} + \frac{1}{16} + \frac{1}{5} = \frac{87}{160} \quad 5
\end{aligned}$$
$\boxed{30}$

(ii)
$$\begin{aligned}
P(M|S) &= \frac{P(M\cap S)}{P(S)} \\
P(M\cap S) &= P(A\cap M\cap S) + P(C\cap M\cap S) \quad 5 \\
&= \frac{9}{32} + \frac{1}{4}\times\frac{3}{5}\times\frac{4}{5} \\
&= \frac{321}{800} \quad 5
\end{aligned}$$

$\therefore P(M|S) = \dfrac{P(M\cap S)}{P(S)} = \dfrac{\cancel{321}^{\,107}}{\cancel{800}_{\,5}} \times \dfrac{\cancel{160}}{\cancel{87}_{\,29}} = \dfrac{107}{145}$ $\boxed{10}$

[see page twenty

(iii) $P(C \cap M | S') = \dfrac{P(C \cap M \cap S')}{P(S')}$

$P(C \cap M \cap S') = P(C) \times P(M|C) \times P(S'|M \cap C)$

$= \dfrac{1}{4} \times \dfrac{3}{5} \times \dfrac{1}{5} = \dfrac{3}{100}$ 5

$\therefore$ The required probability $= 1 - P(C \cap M | S')$ 5

$= 1 - \dfrac{\frac{3}{100}}{1 - \frac{87}{160}}$ 5 $= 1 - \dfrac{3 \times 160}{100 \times 73}$

$= \dfrac{682}{730}$ 5

$\boxed{20}$

(b) $u_i = \dfrac{x_i - A}{10}$

Class Mark	f_i	u_i	$f_i u_i$	$f_i u_i^2$
24.5	1	-3	-3	9
34.5	9	-2	-18	36
44.5	35	-1	-35	35
54.5	40	0	0	0
64.5	12	1	12	12
74.5	3	2	6	12
			-38	104
			5	5

(i) $A = 54.5$

$\bar{x} = A + C \dfrac{\sum_{i=1}^{n} f_i u_i}{\sum_{i=1}^{n} f_i} = 54.5 + 10\left(\dfrac{-38}{100}\right)$ 5

$= 54.5 - 3.8$

$= 50.7$ 5

Modal class 49.5 – 59.5

$M_0 = 49.5 + \left(\dfrac{40-35}{(40-35)+(40-12)}\right) \times 10$ 5

$= 49.5 + \dfrac{5}{33} \times 10$

$= 51.02$ 5

S.D $= \sigma = C\sqrt{\dfrac{\sum f_i u_i^2}{\sum f_i} - \left(\dfrac{f_i u_i}{\sum f_i}\right)^2} = 10\sqrt{\dfrac{104}{100} - \left(\dfrac{-38}{100}\right)^2}$ 5

$= 10\sqrt{(1.04 - 0.38^2)}$

$= 9.46$ 5

$\boxed{40}$

[see page twenty one

(ii) The actual mark corresponding to the mark x_i is $(x_i - 3)$

The actual mean $= \overline{x} - 3 = 50.7 - 3.00$

$= 47.7$ 5

The actual mode $M'_0 = m_0 - 3 = 48.02$

Since the variance does not change, the standard deviation also remains the same. 5

The standard deviation $= 9.46$ 5

$\boxed{15}$

(iii) $\overline{x} = \dfrac{100 \times 47.7 \times 50 \times 55}{150} = 50.13$ 5

$$\sigma^2 = \frac{1}{n_1 + n_2}\{n_1\sigma_1^2 + n_2\sigma_2^2 + \frac{n_1 n_2}{n_1 + n_2}(\overline{x}_1 - \overline{x}_2)^2\}$$ 5

$$= \frac{1}{150}\{100 \times 9.46^2 + 50 \times 2.5^2 + \frac{50 \times 100}{150}(7.3)^2\}$$ 5

$= 73.59$

$\sigma = \sqrt{73.59} = 8.58$ 5

$\boxed{20}$
